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ABSTRACT

This introductory calculus book was especially written for the average or below average student. Its primary intent is to give an overview of basic concepts. Written in programmed instruction format, it contains reviews and self-tests. (Average/LS)

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CALCULUS

FOR THE RELUCTANT LEARNER

by

Peter H. Selby

1 January 1972

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Dedicated to those who had as much difficulty finding out what Calculus was all about as the author did.

SE 018 180

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ARE YOU READY?

Your first question (after you have decided you are one of the people described by the title of this book) probably will be: Am I ready for this encounter with calculus?

To help you answer this question for yourself, here are a few of the assumptions the author had to make concerning the state of your mathematical knowledge to guide him in writing:

1. That you have a working knowledge of basic arithmetic.
2. That you have completed a course in basic algebra.
3. That you are familiar with the general procedures for plotting curves on a rectangular coordinate system (this you should have learned from algebra.)
4. That in addition to the above you also are familiar with the following concepts from trigonometry: the sine, cosine, tangent and secant functions; the slope of a curve (or gradient) at a point, as represented by the tangent; and various ways of indicating the relationships between variables, such as f(x), etc.

Entry into a college course in calculus normally will require that you have had a regular course in trigonometry and, preferably, a course in plane analytic geometry as well -- unless you take the latter course concurrently with you calculus course.

To help you further in making a decision as to your readiness you will find below a brief preliminary quiz covering some of the items referred to above. When you have completed this quiz, check your answers against those given. If you find there are some things you thought you knew but didn't, be sure to look these up in a good text-book before starting.

That way you will be sure you are ready.

Now turn to the next page for the Preliminary Quiz.

Preliminary Quiz

1. A constant is a quantity whose value is _____ (word describing the nature of a constant).

2. A variable is a quantity that____(a, b or c).
   (a) keeps changing
   (b) can assume an indefinite number of values in the same problem
   (c) has only one value for any given problem

3. A function is a relationship between two _____.
   variables
   constants
   ratios

4. A variable whose value depends upon the value of another variable is known as a(n)_____ variable.

5. An independent variable determines the value of the related variable.
   true
   false

6. The symbol $<$ means _____.

7. The symbol $>$ means _____.

8. The symbol $\geq$ means _____.

9. The symbol $\cong$ means _____.

10. A secant line is a line that cuts a curve in two points.
    true
    false

11. A tangent line is a straight line that _____.

12. The tangent represents the slope of a line, or of a curve at a particular point. true false

13. In trigonometry, the tangent function is the ratio of the side opposite an angle(in a right triangle) to the _____ side.

14. The expression f(x) means _____.

15. Cartesian coordinates are rectangular coordinates. true false

16. A polynomial is an algebraic expression that has only positive whole numbers for the exponents of the variables. true false

17. Coefficient is the name given to a factor (or group of factors) of a product to describe its relation to the remaining factors. true false

18. The number 3 in the expression n^3 is called the _____.

19. In the expression (3,2), representing the coordinates of a point, what does the number 2 stand for?

20. The curve known as the parabola is an exponential curve. (Although it wouldn't hurt you to look this up if you don't know the answer, you needn't be too concerned as we will explain it in the text.) true false

Turn to the next page to check your answers.

Answers to Preliminary Quiz

1. fixed
2. b; can assume an indefinite number of values in the same problem.
3. variables
4. dependent
5. true
6. less than
7. greater than
8. greater than or equal to
9. approximately equal to
10. true
11. is perpendicular to a radius at the point where it touches a circle
12. true
13. adjacent
14. function of x
15. true
16. true
17. true
18. exponent
19. the value of y, the ordinate
20. true; because the general equation for the parabola is $y^2 = 4px$. Unless you have studied plane analytical geometry you probably didn't know this. So don't count it off if you got it wrong.

INTRODUCTION

First let me set your mind at ease.

This is not a textbook. It contains few of the proofs so dear to the heart of the professional mathematician. And it is a long way from being even a first course in calculus. On the contrary, it is purposely brief. Like the crosshairs in the telescopic sight of a target rifle, locked on the bullseye, it is aimed at just one thing: Helping you to understand the basic concepts upon which calculus is founded. Nothing more.

You will be given explanations (hopefully the kind that explain), examples and, in appropriate places, a few problems to work -- just enough so that you yourself are satisfied you understand the points being discussed.

The purpose of this book is, very simply, to give you a running start on the subject, prevent you from getting left behind in your regular (school) calculus course, and to keep you from taking a fatal wrong step at the beginning because you have misunderstood -- or not understood -- some important concept. (This usually turns out to be the concept of a limit.)

So relax and enjoy it! No one will be breathing down your neck (at least not in this course). We will proceed at a leisurely pace, taking one small step at a time. Hopefully you will be reading this book before starting your college calculus course. But if not before, then at least

as early as possible during the course, as a supplement to your required text.

At the conclusion of the chapters dealing with differential calculus, and again after the chapters on integral calculus, you will find short, self-administered quizzes. Their only purpose is to enable you to check up on yourself and see how well you have done. The results should improve your morale and help remove some of the trepidation that normally accompanies one's first dip into the sea of calculus.

NOTE

Much of this book has been put together differently from most books you have read. On many pages you will be asked a question or asked to supply an answer. When this happens, turn to the page indicated to check your answer and to continue.

Because not all of the pages are intended to be read consecutively, it would be helpful to use a bookmark to help you to keep your place.

Now turn to page 1, please, and we'll get started.

CHAPTER 1: WHY CALCULUS?

This is a question you deserve to have answered at the outset.

If you have decided to study the subject then presumably you have some reason for doing so. Is it because it is a required course for some branch of engineering? Psychology? Chemistry? Physics? Economics? Advanced basket weaving? If so there is nothing wrong with that as a reason, except that it would be helpful to know *why* it is a required course.

Are you studying calculus because you plan a career in mathematics? If so, then I am sure you will need little convincing as to the importance of acquainting yourself with this powerful mathematical tool. Doubtless you will have discovered already that calculus is, purely and simply, the starting point of all advanced mathematics. Without it you are stopped before you start. Without a working knowledge of arithmetic you would not get far with algebra. Without a working knowledge of algebra you will not get far with calculus. And without calculus you would not get far with *any* aspect of mathematics beyond the elementary.

Why is this so? Because calculus enables the solution of problems that cannot be solved in any other way. And even problems that can be solved in other ways often can be solved faster, more accurately, or both with the aid of calculus.

For example. How would you go about finding the speed, at any exact instant of time, of a baseball thrown up into the air or dropped from a tall building? Remember, it will be acted upon by gravity all the time it is in the air. Gravity will constantly be trying to pull the ball back to the earth's surface. The effect on a ball thrown up into the air will be to slow its passage upward and to accelerate its return to earth. Thus its speed will be changing constantly under the attractive force of gravity.

It would be possible, of course, to <u>estimate</u> the ball's actual speed at any particular moment by calculating its <u>average</u> speed during a very short period just before and just after the selected moment. (We will do this a little later on so that you can see the method.) But this would yield only an approximation of the ball's <u>instantaneous</u> speed, that is, its speed at any specified instant of time.

The fact of the matter is that before the advent of calculus there was no way to compute instantaneous speeds of this nature either quickly nor accurately. Mathematicians simply didn't know how to do it; they did not have the necessary mathematical tools. And this was most frustrating to them because around the beginning of the 17th century (as you will learn from Chapter 8) the natural scientists of that day were studying the movement of pendulums, the planets, and all kinds of moving objects quite intensively. The fact that many such objects moved at varying speeds gave rise to the question from which calculus was born, namely: "What is speed?"

The question of speed is, then, the fundamental one of calculus. But this does not mean that the use of calculus is confined solely to the study of falling objects. the movement of planets or purely mechanical matters. Another name for speed is rate of change, and the question of how to determine rates of change occurs in many different situations. Thus, we find calculus applied to all aspects of physics -- heat, light, sound, magnetism, electricity, gravitation, the flow of water. Calculus enabled James Clerk Maxwell to predict radio twenty years before any physicist could deminstrate radio experimentally. Einstein's theory of 1916 and the atomic theories of the nineteen-twenties relied heavily upon calculus.

In addition to these applied aspects of calculus it also stimulated the development of several new branches of pure mathematics. In fact, few branches of mathematics have appeared in this century that do not use calculus. Anyone attempting to study these subjects without a background in calculus would be lost. Problems that can be handled relatively simply with the aid of calculus become enormously difficult to solve -- if indeed they can be solved -- without it.

In addition to making it possible to handle dynamic problems, such as the relationship between time, speed and distance, the development of calculus also turned out to supply a method for analyzing curves. The subject of curves may seem a bit remote from that of speed. How-

ever, interestingly enough, the problem of finding the rate of change of direction of a curve at a given point (which is, of course, measured by the rate of change of the slope of the curve at the point) is closely related to the physical problem of finding the instantaneous speed of a moving body. We will look into this matter a little later on in the book.

Now although calculus grew from a fairly simple idea, the idea of speed, there is a general impression that it is a difficult and complex subject. And so it is -- or can be. Its difficulty and complexity depend upon how far into the subject one attempts to go without proper preparation. The fundamental concepts are quite simple, and these are all we will attempt to cover in this short treatment of the subject. But there is practically no limit to how far into it one can go or how many applications and new fields for exploration one can find by studying it. The housewife who finds an electric egg-beater easy to operate might experience some difficulty solving the mechanical and electrical problems involved in designing it.

Similarly, most people can, with a little effort, learn how to apply calculus successfully in solving many kinds of practical problems. However, few would find it easy to devise new applications or to apply it in the more abstract and theoretical aspects of advanced mathematics.

How much of calculus should we, then, attempt to cover in an introductory book?

I believe the answer is: (1) only what the reader needs to know in order to be able to use the proccesses of differentiation and integration where appropriate; (2) enough so that he can recognize at least some situations where its use is appropriate; and (3) sufficient theory (that is, familiarization with the concepts upon which calculus is based) to assist his passage through a formal (college) course.

Obviously this is the kind of compromise that will please no one except, hopefully you -- the learner. But then this book was written for you, so we need not concern ourselves with any alarmed denouncements by the professional mathematicians. Dissatisfaction or failure to learn on your part is, however, another matter. That we are very much concerned about.

Chapter 8, to which we already have referred, contains a brief historical review of the development of mathematics and of the kinds of problems that early mathematicians found it difficult or impossible to solve until the advent of calculus. It has purposely placed at the rear of the book so that you may read it or not, depending upon the extent of your interest in how calculus came to be.

Let us now proceed to Chapter 2 where we will begin our investigation into the rudimental aspects of calculus, working our way slowly, step by step, from the presently known to the presently unknown.

CHAPTER 2: SO WHAT'S NEW ABOUT LIMITS?

Let's begin with what you already know about limits.

Did you ever feel you were reaching the "limit of your patience?" This thought is based on the notion (which we won't debate now) that each of us has only a fixed supply of patience and that circumstances can make one feel he has just about used up his supply. A mathematical way of saying this would be to say that our reserve (remaining amount) of patience is approaching zero as a limit. And using standard mathematical symbols we could express this situation symbolically as: Patience $\longrightarrow 0$. (In case you have forgotten, the arrow means "approaches.")

Similarly, when we speak of reaching the "limit of our endurance" we really are referring to the fact that our supply of energy is fast approaching zero as a limit. Thus: Endurance $\longrightarrow 0$.

Many of us have been faced with the dilemma of having the amount of gasoline remaining in our gas tank "approach zero" at an inopportune moment. We also know about military limits (being "off limits"), speed limits, the ground being the limit for a falling ball, etc.

These examples all have something in common. Can you tell what it is? Try putting it into words, then check your answer with the one given.

Turn to page 8 to check your answer.

ans. The concept of the distance from some fixed position, or of a quantity, approaching zero as a limit.

The foregoing examples are fine for developing an intuitive notion of limits. But in order to be able to use this concept to help solve the kinds of problems that concerned Newton, Leibnitz and most of the other early mathematicians back to the days of the Greeks, we will need to examine it more closely.

Notice that, for the most part, we tend to think in terms of the portion of the original amount remaining, rather than in terms of how near that remainder is to zero. Thus, we are more concerned with the amount of patience used up rather than with how much remains; how fast we are travelling rather than how much our speed differs from the speed limit; the amount of gas left in the tank rather than how close this amount is to zero. It is a subtle attitude of mind or way of thinking that we need to become aware of in our discussion of limits.

The difference between our intuitive concept of limits and the mathematical concept lies in this important fact: In mathematics we are primarily interested in the difference between some amount and the limit zero.

Consider this idea with relation to a specific speed limit such as 35 mph. Usually we would say that as our car speed increases, it approaches 35 mph as a limit. How could you express this situation in terms of the difference between your speed and 35 mph?

Check your answer on page 10.

ans. as $S_c \longrightarrow 35$, $D_s \longrightarrow 0$

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In the answer given above S_c was used to represent the speed of the car as it approached 35 mph and D_s stood for the difference between the speed of the car and the speed limit of 35 mph (other symbols would serve as well).

No doubt you could think of many other examples, but even these few are sufficient to allow us to arrive at some kind of a general statement about such situations.

We might, for example, say something like this: As the value of any quantity approaches some limit, the difference between the value and its limit approaches zero. Symbolically expressed it would look something like this:

$$\text{as } V_q \longrightarrow L, \quad (V \sim L) \longrightarrow 0.$$

V_q stands for the value of the quantity (whatever its nature -- gallons, miles per hour, inches, oranges, light years), L stands for the limit the value is approaching, and $V \sim L$ represents the amount by which the value differs from its limit at any given moment.

Now suppose you were climbing a mountain and your objective was to reach a height of 5000 feet above sea level. If we let h represent your height above sea level and L represent your altitude goal, how would you interpret in words the following symbolical representation -- or mathematical model -- of this situation?

$$h \longrightarrow L, \quad (h \sim L) \longrightarrow 0$$

Turn to page 11.

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ans. As your speed increases, the difference between your speed and 35 mph approaches zero as a limit.

Do you see the difference?

As we commonly use the word "limit" we are chiefly interested in the magnitude or size of a quantity as it gets nearer and nearer to some limit. In mathematics we are more interested in the difference in or distance between the quantity and its limit.

Relating these two notions we can say that as a quantity approaches a limit, the difference between the quantity and its limit approaches zero as a limit.

In order to get used to seeing what this kind of relationship looks like in mathematical shorthand, let's try expressing it symbolically. Take the case of the filling gas tank. As the quantity of gasoline in the tank approaches 16 gallons (the tank's capacity), the space remaining in the tank approaches zero. We can express this as follows:

$$\text{as } Q_g \longrightarrow 16, \quad S_r \longrightarrow 0,$$

where Q_g represents the quantity of gas (in gallons) and S_r represents the space remaining. Obviously all we have done is to use the little arrows to mean "approaches" and invented a few letter symbol to represent the values involved. Not very technical and certainly not very formal mathematics, but it says what we want it to say and that's the only purpose of any mathematical symbol.

Now suppose you make up some symbols of your own and try representing the situation where your car speed is approaching the posted speed limits of 35 mph. When you have something that looks right to you, check it against the symbology shown on page 9.

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ans. As your height above sea level approaches the goal (limit), the difference between your height and your goal (limit) approaches zero.

Your interpretation should have been generally similar to that shown above. We could, of course, have used 5000 (feet) in place of the symbol L (for limit) since we happen to know the numerical value of the limit in this instance, in which case we would write

$$h \longrightarrow 5000, \quad (h \sim 5000) \longrightarrow 0.$$

If you are saying to yourself that these examples are absurdly simple, you are quite right. However, I urge you to remember this with gratitude when we get to some that are not quite so obvious.

Now it's time to apply the notion of limits to a practical problem in order to discover whether or not it really helps solve (or makes possible the solution of) the problem. We are going to approach it in a slow, relaxed way so don't panic.

The problem we will consider actually is the basic one that prodded Sir Isaac Newton into developing his method of "fluxions" which, as mentioned earlier, later became known as Calculus (or, more elegantly, the Calculus). Simply stated, what puzzled Newton (among many other things, no doubt) was how to determine the instantaneous velocity of a freely falling body -- discounting the resistance of the air through which it was passing.

Sound simple?

Turn to page 12 and let's see.

A ball thrown into the air is an example of a body under the influence of gravity, so let us suppose we throw a baseball straight up. What happens to it?

We know from experience that (depending upon how hard we throw it) the ball will go up for a certain distance, then fall down to the ground, from which it started. Since the effect of gravity is to "pull" objects down, it is perfectly evident that while the ball is moving upward the effect of gravity will be to reduce its speed continuously until it finally reaches some maximum height, changes direction, and begins its return trip to earth. On the way down gravity will cause it to speed up as it approaches the gound. In other words, our own experience tells us that the ball we throw upward is going to be changing velocity* all the time it is in the air! Our job will be to determine its speed* at any given instant, that is, its instantaneous speed. To do so we will use the concept of "approaching a limit" which we have been discussing.

Where do we start?

It is not too difficult to determine the ball's height above the ground at any instant (assuming one knows the initial conditions). Physicists have done it with great accuracy and derived an equation that represents the height of the ball with time.

Suppose, therefore, we are given the information that the expression $h = 128t - 16t^2$ is the relation between

*Mathematicians and physicists make a distinction between the terms "speed" and "velocity," however for our purposes we will consider them synonymous.

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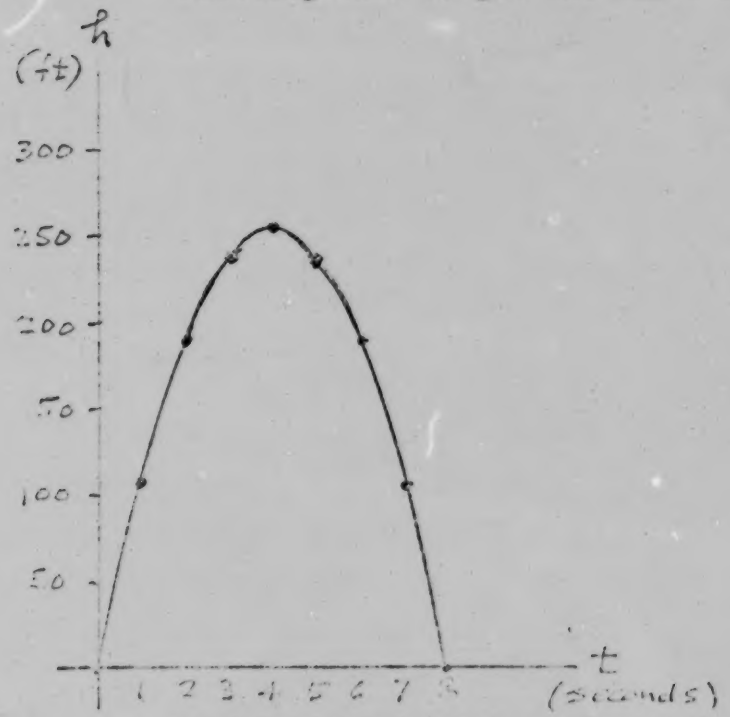
h (height) and t (time in seconds) for as long as the ball is in the air. Let's see how this equation will help us learn something about the velocity of the ball.

Here is the equation again: $h = 128t - 16t^2$

Since h and t are related variables let us assign a series of values to t, as the independent variable, and see what values we get for h. As shown in the table below we find that as t increases in value from zero to 4 seconds, h increases also. However, once we get <u>beyond</u> 4 seconds, for successive values of t we find that h <u>decreases</u>, until finally, at t = 8 seconds, h becomes zero.

Plotting this we get the curve shown below.



t	h
0	0
1	112
2	192
3	240
4	256
5	240
6	192
7	112
8	0

What was the ball's <u>average</u> velocity during the first four seconds?

Turn to page 14 to check your answer.

ans. $\frac{256}{4}$ = 64 feet per second

Remember: Average velocity is always found by dividing the change in position of a body (that is, the distance moved) during some period of time, by the elapsed time. And since the ball we threw up into the air traveled a distance (height) of 256 feet in four seconds, its average velocity was 256 divided by 4, or 64 feet per second (fps).

Notice, however, that its actual speed changed during those four seconds. During the first second it traveled 112 feet, hence its average speed between zero and one second was 112 feet per second. During the next second the ball traveled a distance of 80 feet (192 minus 112), hence its average speed during that second was only 80 feet per second. Similarly, its average speed between the second and third seconds was 48 feet per second, and between the third and fourth seconds only 16 feet per second. Obviously it was slowing down in a hurry.

Now this is all very interesting as it relates to how a ball changes velocity under the influence of gravity. However, it doesn't answer our original question, namely: How do we determine the ball's exact speed at any given instant, as distinguished from its average speed?

Let us suppose, for example, we wish to know the ball's instantaneous velocity when t = 2 seconds. How would we go about finding it? Give it some thought and then check your answer against the one given on page 16.

ans. $h = 128t - 16t^2$

Very well. Using this equation ($h = 128t - 16t^2$) allows us to calculate the following values for h for the selected values of t:

at t = 1.99 seconds, h = 191.3584 feet

at t = 2.00 seconds, h = 192.0000 feet

at t = 2.01 seconds, h = 192.6834 feet

Therefore, during the one-hundredth of a second before t = 2, the ball traveled 192.0000 - 191.3584 or 0.6416 feet, and this distance divided by one-hundredth of a second gives us an average speed of 64.16 feet per second. Similarly, during the hundredth of a second after t = 2, the ball traveled 192.6834 - 192.0000 or 0.6384 feet, which is equivalent to 63.84 feet per second.

Adding the before and after speeds (64.16 + 63.84) and dividing the sum by 2 we arrive at a "guesstimate" of 64 feet per second as the ball's instantaneous velocity at t = 2. In other words, the ball's speed at the instant t = 2 appears to be approaching 64 feet per second as a limit!

In order to help you see this a bit more clearly we have drawn a graph of the situation, shown on page 17.

Turn to page 17.

ans. We can do it by calculating <u>average</u> velocities over shorter and shorter periods of time, both before and after t = 2.

I hope your answer (if you were able to come up with one) was something like the one given above because this is a very important point. Let's consider it for a moment.

Remember: We don't as yet have any <u>direct</u> way of calculating the <u>instantaneous</u> velocity of the ball at a given point in time, such as t = 2 seconds. So far, the only way we know to figure its speed is to compute its <u>average</u> speed during some period of time very close to t = 2 seconds. The closest to t = 2 would be some very small fraction of a second <u>just before</u> or <u>just after</u> t = 2 seconds.

Now since we know that its actual velocity is changing constantly (remember that the ball is constantly accelerating or decelerating under the force of gravity), its average speed just before and just after t = 2 will be different, however slight that difference. But if we take a time interval sufficiently small -- say, one-hundredth of a second -- before and after t = 2 to calculate its average speed, it seems reasonable to assume that the ball's instantaneous speed at t = 2 should be about midway between these two values.

To perform these calculations we will need our equation giving the change in altitude (distance) with time. Do you recall what that equation is? See if you can write it down in the space provided below:

_____

Turn to page 15.

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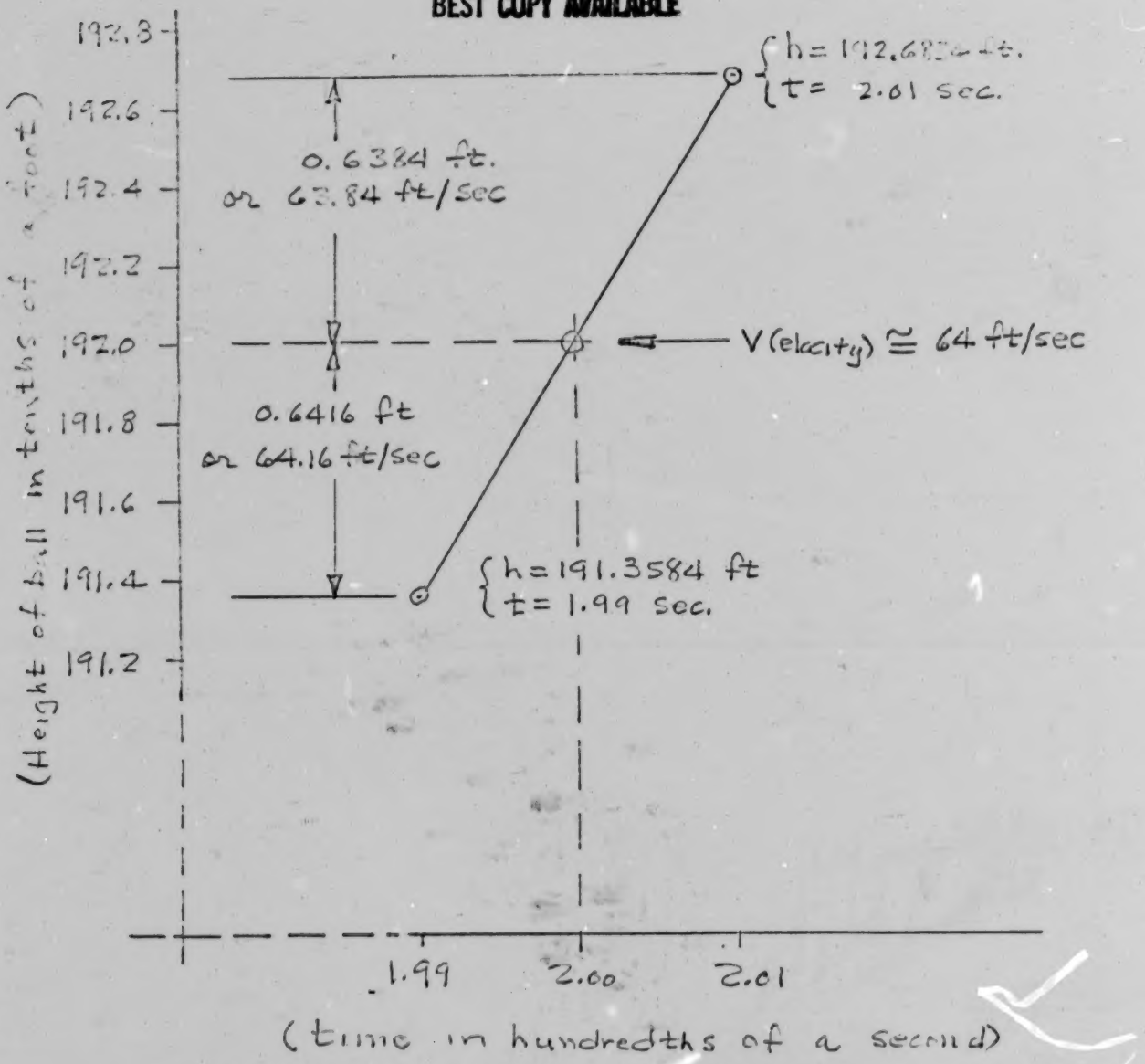

Plotting the two values of h for t = 1.99 and t = 2.01 and connecting these with a straight line we get the time-altitide graph shown above.

It is again apparent that the velocity of the ball <u>just before</u> t = 2 was slightly <u>greater</u> than 64 fps, and <u>just after</u> t = 2 it was slightly <u>less</u> than 64 fps, which tends to confirm our suspicions that at t = 2 its speed was very close to 64 fps.

Now, did we just do something we said couldn't be done without calculus? Did we, that is, find the instantaneous velocity of the ball by algebraic methods? The answer is No to both questions. Why?

First, although we did find its velocity <u>just before</u> and <u>just after</u> the selected instant of time t = 2 seconds, we did <u>not</u> find it for the exact instant t = 2! True, it <u>appears</u> that its speed at t = 2 is approaching 64 fps as a limit (that is, not more nor less than 64 fps), but this is not proof. It is an assumption on our part. And even if we took smaller and smaller intervals of time both before and after t = 2 in which to calculate the ball's average velocity, it still would be just that -- an <u>average velocity</u>* taken over a period of time, however short -- and not an <u>instantaneous velocity</u>.

And second, if we were willing to be content with assumptions of this kind, it certainly is a long, hard way to find even an approximate answer.

Both Newton and Leibnitz, plagued by this and similar problems, felt there had to be a better way. And thanks to their persistence and brilliant thinking they found that way.

So limber up <u>your</u> thinking, pay close attention, and we'll retrace the line of reasoning they went through (each in his own way, actually, and independently of one another)

---

*Just a reminder of our definition on page 14 of average velocity as the distance traveled between two points in time, divided by the time interval.

in discovering a beautifully simple way to find such things as instantaneous rate-change values. We will find out how they learned to apply, in a precise way, the method of limits which we were able only to approximate (and that laboriously) in the foregoing example.

CHAPTER 3: AN EASIER WAY TO SOLVE RATE PROBLEMS

We are going to examine, step by step, the relationship between two variables as one changes with respect to the other -- and particularly the rate of this change. For this purpose we will graph the situation; this will help us visualize it better and also aid in considering it algebraically.

We could use as an example the relationship $h = 128t - 16t^2$, the change of height with time of the ball thrown in the air, discussed in the last chapter. Here we were concerned with the two variables h and t, where h was expressed as a function of t, the independent variable. We found, as you will recall, that because the ball's height did not change uniformly with time, its velocity was not constant. Hence the task of finding its velocity at any given instant was not an easy one. We will return to this problem a little later on. But for our present discussion we will use the even simpler relationship between two variables, $y = x^2$.

Do you remember from algebra what we call this kind of a function? If so, write it in the space provided below.

_____

Turn to page 22 to check your answer.

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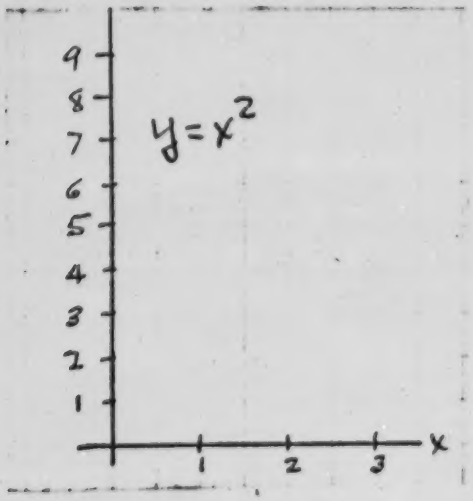
ans. A power function, or square function.

Yes; it is usually termed a "power function" for the obvious reason that the independent variable (x) appears at a higher power than one. If you have studied any plane analytical geometry you may also recognize $y = x^2$ as the equation for the curve known as a parabola.

Using the values for x shown in the table at the right, find the corresponding values for y and plot the resulting curve on the coordinate system provided below. Then check your results with those shown on page 24.

y	x
	0
	1
	2
	3



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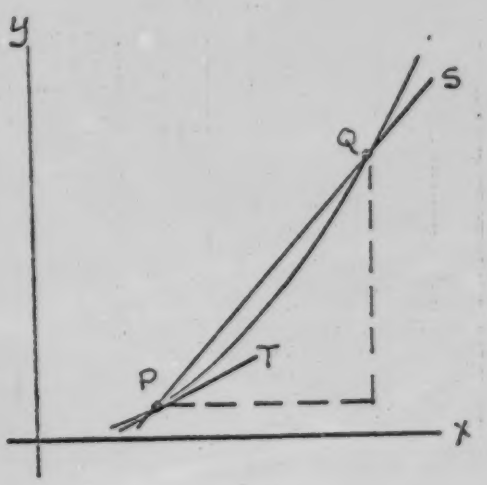
ans. T = tangent to curve at point P = $\frac{a}{b}$ = slope of tangent line T.

Thus we have found that it is convenient to represent the slope of the curve at the point P by means of a line, T, tangent to the curve at that point.

Now to assist us in our analysis let's add another, random point on the curve at some indeterminate distance from P. This point we will designate as Q.

Connecting this point to P by a straight line gives us the secant line, S. Observe this in the figure below.

(Note: In case you have forgotten, a secant may be defined as a straight line that cuts a curve at two points.)



How should we designate the coordinates of the point P, bearing in mind that P is <u>any</u> point on the curve?

Turn to page 25.

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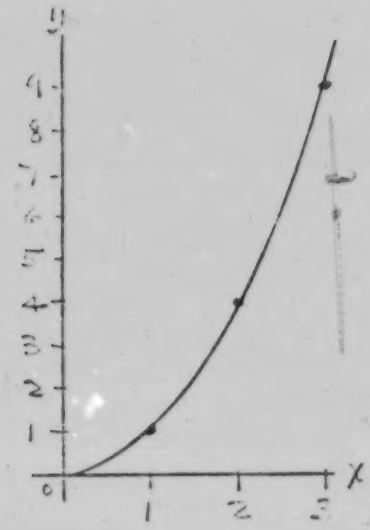
ans.



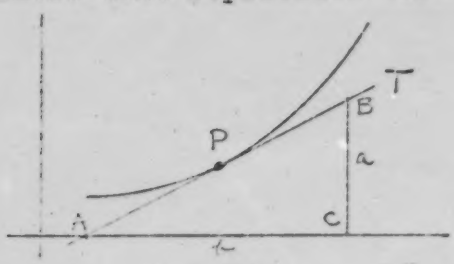
y	x
0	0
1	1
4	2
9	3

Note from your curve that, just as the velocity of the ball thrown into the air was constantly changing, so the direction of the curve is constantly changing, reflecting the rate at which one variable is changing with respect to the other. So if we can find some way to determine the instantaneous rate of change of direction at any point on the curve, we should be able to use the same general approach to find the instantaneous rate of change in the velocity of the ball. Why? Because although the variables are different in each case and the physical situations they symbolize are different, mathematically the two equations involved are essentially similar in nature!

But how do we find the direction of the curve?

Well, the direction of a curve at any point is, as you may recall, simply the slope of the curve at that point. Therefore, what we really are seeking is the slope, or angle of inclination, between the (positive direction of) the x-axis and a line tangent to the curve at the given point.

In the sketch below, identify the line T and give the ratio that represents the slope of T.



P = POINT ON CURVE
T = ?
? = SLOPE OF T

Turn to page 23 to check your answers.

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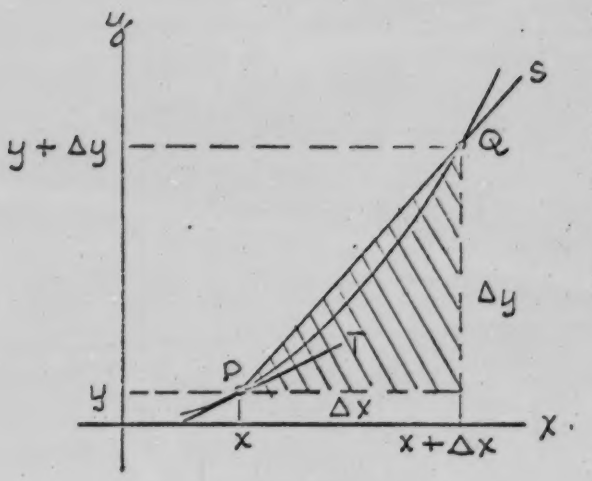
ans. It probably would be best to designate the coordinates of P as (x,y) in order to illustrate the general nature of this point.

We also need to indicate the position of the point Q with relation to P. And since Q is a bit further from the x-axis and y-axis than P, we designate the horizontal distance from Q from P as Δx (<u>delta x</u>, that is, a little bit of x), and the vertical distance as Δy (<u>delta y</u>, that is, a little bit of y).

With this information added our graph now looks like this:



Coordinates of P are: (x,y)
Coordinates of Q are:
(x + Δx), (y + Δy)

Try writing the equation for the <u>slope</u> of the secant line S, keeping in mind that it will simply be the ratio of the vertical distance to the horizontal distance between the points P and Q.

Turn to page 26.

ans. Slope of S = $\frac{\Delta y}{\Delta x}$

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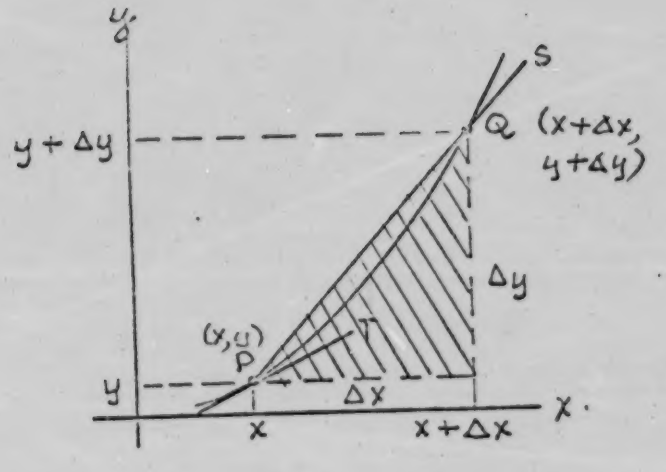

Slope of S = $\frac{\Delta y}{\Delta x}$

It may appear as though we had only succeeded in accumulating an odd assortment of letters. However, don't be alarmed. They are all necessary and will be of great help shortly. You will note also that we have shaded in the triangle of which the secant line S is the hypotenuse. This was done to help focus your attention on it.

Now, remembering that the equation for our curve is $y = x^2$, substitute the coordinates of the point Q for x and y in this equation and see what kind of an expression you get.

Turn to page 28 to check your answer.

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ans. $(x+\Delta x)^2 = x^2 + 2x\cdot\Delta x + \overline{\Delta x}^2$ (the little bar, or vinculum, over the Δx means that the exponent applies to the entire expression, not just to the x).

What we are seeking by this algebraic procedure is a relationship between Δx and Δy. Specifically, what we would like to find is the ratio of Δy to Δx (that is, the slope of the secant line S) based on what we know about the equation for the curve. Once we find this, you will see how we plan to use it.

So, from the previous page we now have this information:

$$y = x^2 \qquad (1)$$

and

$$y + \Delta y = x^2 + 2x\cdot\Delta x + \overline{\Delta x}^2. \qquad (2)$$

But since from (1) we know the value of y in terms of x, we can substitute x^2 for y in equation (2) and get:

Turn to page 29.

BEST COPY AVAILABLE

ans. You should get $(y + \triangle y) = (x + \triangle x)^2$

Once more.

The equation for the curve is $y = x^2$.

The coordinates of the point are:

x coordinate = $x + \triangle x$,

y coordinate = $y + \triangle y$.

Substituting these coordinates in the equation of the curve gives us

$$(y + \triangle y) = (x + \triangle x)^2$$

Your next step will be to expand the binominal $(x + \triangle x)^2$.

Do so, and write your answer in the space below.

$$(x + \triangle x)^2 = x^2 + _____$$

Turn to page 27 to check your answer.

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ans. $x^2 + \Delta y = x^2 + 2x \cdot \Delta x + \overline{\Delta x}^2$

or, subtracting x^2 from both sides,

$$\Delta y = 2x \cdot \Delta x + \overline{\Delta x}^2$$

Thus we now have:

$$\Delta y = 2x \cdot \Delta x + \overline{\Delta x}^2$$

And dividing each term on both sides of the equation by Δx gives us

$$\frac{\Delta y}{\Delta x} = 2x + \Delta x$$

This looks a bit simpler, doesn't it?

But what does it represent?

See if you can complete the following sentence:

The quantity $2x + \Delta x$ represents_____

_____

Turn to page 30 to verify your answer.

ans. the slope of the secant line s.

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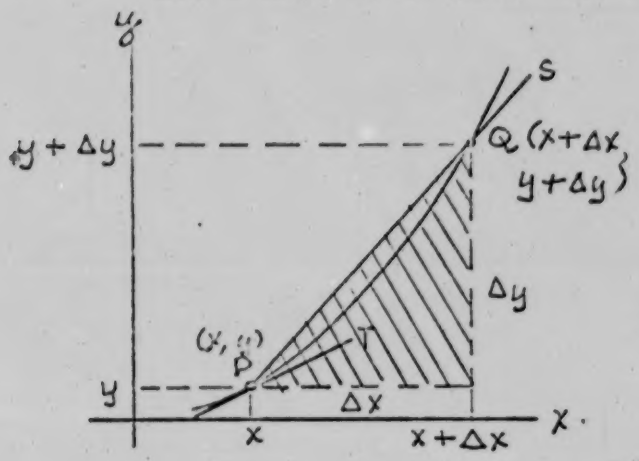
I hope you got it right!

Here it is again:

$\frac{\Delta y}{\Delta x} = 2x + \Delta x$ = slope of the secant line S.

Let's look at it once more in the graph below.



$\frac{\Delta y}{\Delta x} = 2x + \Delta x$

Think about the secant line S again for a moment. What it really represents, in effect, is the average slope of the curve between the two points P and Q, in much the same way as the velocity value we found for the ball between any two instants of time represented the <u>average velocity</u> of the ball. But just as we were seeking the instantaneous velocity there, here we are seeking the exact slope of the curve at a specific point -- not the <u>average</u> slope.

Continue on page 31.

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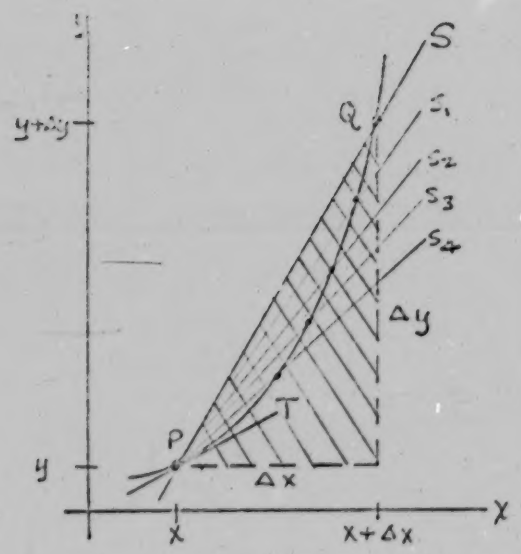

Slope of S

$$\frac{\Delta y}{\Delta x} = 2x + \Delta x$$

Very well then. Since what we <u>really</u> want is the slope of the curve $y = x^2$ at the precise point P, let us imagine the point Q to move slowly along the curve towards P. What we now get is a series of secants (shown above as S_1, S_2, S_3, S_4, etc.).

At the same time -- since they are associated with (<u>define</u>, actually) the position of the point Q -- the distances Δy and Δx grow shorter and shorter and our shaded triangle diminishes in size.

Obviously Q is approaching a <u>limit</u> (sound familiar?), namely, the point P.

What limit is the secant S approaching?

Turn to page 32 to check your answer.

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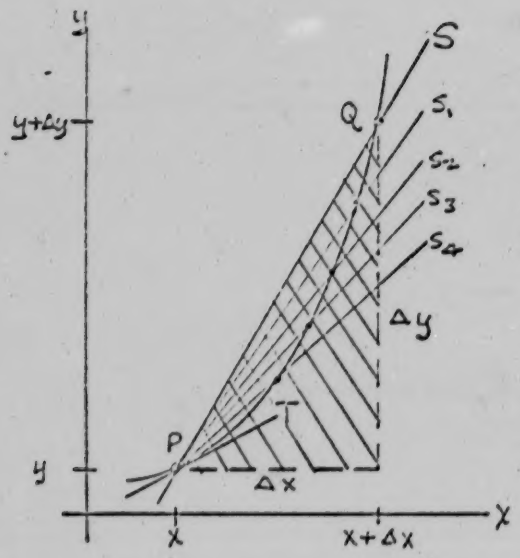
ans. The tangent line T.



Of course; the secant S is approaching the tangent line T as a limit. By the time the point Q reaches point P, the secant S (one end of which moves with Q) will coincide with the tangent, T. Not only <u>coincide</u> with it; it will <u>become</u> the tangent of the curve at the point P.

It is important that you see these two things very clearly:

1. Q is approaching the point P <u>as a limit</u>!
2. The horizontal distance, Δx, between Q and P, is approaching <u>zero</u> as a limit.

How do you think the expression for the slope of the secant, $\frac{\Delta y}{\Delta x} = 2x + \Delta x$, will change as Δx approaches zero as a limit?

_____

_____

_____

Turn to page 34.

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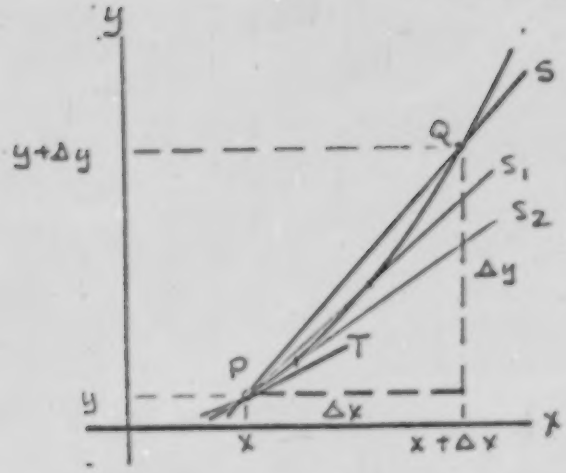

To summarize, then:

1. As Q approaches P as a limit, and
2. Δx approaches zero as a limit, then
3. The <u>secant</u> tends to become <u>tangent</u> to, and therefore the slope of, the curve at the point P!

Using the arrow (symbol for "approaches") which we used earlier, and the abbreviation "lim" for <u>limit</u>, we can express symbolically what is happening like this:

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 2x$$

Try putting this symbolical expression into words just to make sure you understand its meaning.

_____

_____

Turn to page 35 to check your answer.

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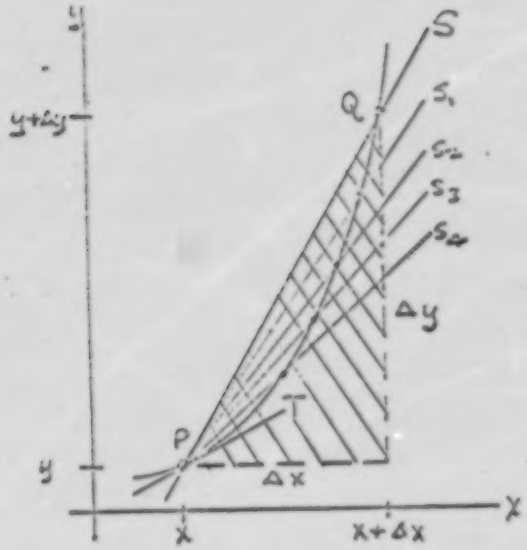
ans. The term Δx will drop out of the right-hand member of the equation, leaving just the term 2x.



True. But if Δx, approaching zero, becomes so infinitely small that is in effect drops out of the right-hand member of the equation $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 2x + \Delta x$, leaving just 2x, it seems reasonable to ask, Why doesn't it also drop out of the expression $\frac{\Delta y}{\Delta x}$ on the left-hand side?

The answer is: Because although Δx is approaching zero as a limit, so is Δy. Hence (to oversimplify a matter that involves the theorems of infinitesimals), the ratio $\frac{\Delta y}{\Delta x}$ remains intact.

Remember: $\frac{\Delta y}{\Delta x}$, interpreted graphically, is approaching the tangent to the curve at the point P. This is a <u>specific</u> number of value! So while Δx is approaching zero, $\frac{\Delta y}{\Delta x}$ is approaching a <u>real</u> value, namely, the slope of the curve at the point P. Therefore the diminishing value of Δx has a different effect on the two sides of the equation.

Turn to page 33.

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ans. The limit of $\frac{\Delta y}{\Delta x}$ as Δx approaches zero equals 2x.

Let's repeat the entire limit formula so we'll have it in front of us:

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 2x,$$

or, put into words, as Δx approaches zero as a limit, the limit of the ratio $\frac{\Delta y}{\Delta x}$ in the expression $\frac{\Delta y}{\Delta x} = 2x + \Delta x$ becomes 2x.

Now, what have we really discovered from all this investigating that we didn't know when we started out -- and that is useful? It is important that you know before going on, so see if you can select the <u>best</u> answer below.

1. The secant becomes the tangent as Δx approaches zero as a limit.
2. As the interval Δx of the independent variable approaches the limit zero, the ratio $\frac{\Delta y}{\Delta x}$ becomes the instantaneous rate of change (or growth rate) of the function $y = x^2$ at the point P.
3. In the expression $\frac{\Delta y}{\Delta x} = 2x + \Delta x$, the term Δx drops out as Δx approaches zero as a limit.

Turn to page 36 to check your answer.

ans. No. 2.

Answers 1 and 3 both are correct statements, but neither is the best answer nor the most significant thing that occurs.

The really important piece of information is that we have found an expression for the instantaneous rate of change of the curve -- that is, of the function which the curve represents -- at a specific point, or instant!

To understand the real significance of this, realize that if $y = x^2$ happened to represent the relationship between the height and time increments of the ball thrown into the air, then 2x would represent the instantaneous velocity (time rate of change) of the ball at any given moment! Exactly what we were trying to find!

In other words, we have essentially done what we set out to do, namely, discovered a means of calculating instantaneous rate of change, or growth rate, of a function at a given instant.

Just to prove we've done it, we're going to go back to our equation for the thrown ball in a moment and use it to obtain the instantaneous velocity of the ball. But first let's have a short review.

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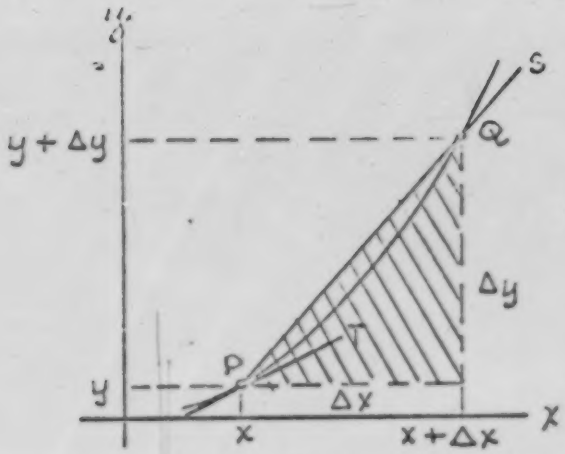

The above picture should look quite familiar to you by now. Here then are the steps we went through in finding the derivative (derived function, or instantaneous rate of change) of y with respect to x.

Given function (curve)	:	$y = x^2$
Coordinates of curve at Q	:	$(x + \Delta x)$ and $(y + \Delta y)$
Substituting these values in the given function we get	:	$y + \Delta y = (x + \Delta x)^2$
And expanding the right member	:	$y + \Delta y = x^2 + 2x \cdot \Delta x + \overline{\Delta x}^2$
Substituting x^2 for y (from our original equation) we get	:	$x^2 + \Delta y = x^2 + 2x \cdot \Delta x + \overline{\Delta x}^2$
Subtracting x^2 from both sides	:	$\Delta y = 2x \cdot \Delta x + \overline{\Delta x}^2$
And dividing both sides by Δx	:	$\frac{\Delta y}{\Delta x} = 2x + \Delta x$
Finally, taking the limit of the function as $\Delta x \longrightarrow 0$ (that is, as the point Q approaches point P on our graph) we get	:	$\lim_{\Delta x \longrightarrow 0} \frac{\Delta y}{\Delta x} = 2x$ = the "derivative" of y with respect to x for the function $y = x^2$, or the instantaneous rate of change of y with respect to x.

Now turn to the next page where we will discuss a final important concept about the limiting value of the function as $\Delta x \longrightarrow 0$.

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The important concept we referred to on the preceding page is this:

There is a special name for the limiting value of the ratio $\frac{\Delta y}{\Delta x}$ as Δx approaches the limit zero. That name is "derivative" (did you notice where we used it on page 37 without explanation?). It is written as $\frac{dy}{dx}$ and read as "dee-wy, dee-eks."

In other words (or symbols), a derivative = $\frac{dy}{dx}$ = $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$. $\frac{dy}{dx}$ is, then, the <u>limiting</u> value of the ratio $\frac{\Delta y}{\Delta x}$.

<u>Another way of saying this is to say that $\frac{dy}{dx}$ is the customary expression for the derivative of y with respect to x.</u>

You have now been initiated into some of the mystical language of calculus and can use the term "derivative" to astound your friends.

Seriously, and more importantly, you have been exposed to what is probably the most fundamental concept in differential calculus: The concept of the derivative of a function.

In the next chapter we are going to look at some of the ways in which this concept is applied to functions of various kinds.

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CHAPTER 4: SOME HANDY SHORTCUTS

Together we have worked our way through the "delta process" of finding the derivative of a function -- not <u>any</u> function, but the specific function $y = x^2$.

We found, for example, that by this method of successive approximation -- algebraically derived -- we arrived at an expression for the instantaneous rate of change of one variable (y) with respect to another variable (x).

Thus the <u>limit</u> of the rate of change of y with respect to x in the expression $y = x^2$ was shown to be

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = 2x.$$

If we were to write this in the form of an <u>instruction</u> rather than a <u>result</u> would you know what to do with it? Let's find out.

Complete the following:

$$\frac{dy}{dx}(y=x^2) = _____$$

Turn to page 40 to check your answer.

ans. 2x

I hope you figured it out for yourself. Here it is:

$$\frac{dy}{dx}(y = x^2) = 2x.$$

We read this as: The derivative of y with respect to x in the expression $y = x^2$ is 2x. Other symbols used to denote the derivative of y with respect to x are: y', f'(x), $\frac{d}{dx}y$, and $\frac{d}{dx}f(x)$. f(x) means, of course, the function (<u>any</u> function) of x.

For example, in the expression $y = x^2$, y is the function of x. Hence we could write this as $f(x) = x^2$. However, we usually use y to represent the dependent variable since it is helpful to graph many of these functions and using y with x provides the two coordinates necessary for plotting in our familiar Cartesian coordinate system.

Now let's consider the matter of finding the derivative of an expression strictly by algebraic means.

Look again at our original function, $y = x^2$, and its derivative, y' = 2x (using one of the new and convenient notations for the derivative). How could we have manipulated the term x^2, mathematically, in order to turn it into 2x? That is, what would we have had to <u>do</u> to it?

Turn to page 42 to check your answer.

ans. $y' = 3x^2$

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Did you get it right?

Let's do it together to make sure.

Following our empirical rule we multiply the coefficient of x^3 by the exponent 3, at the same time subtracting 1 from the exponent. This gives us

$$y' = 3 \cdot 1\ x^{3-1}\ , \text{ or } y' = 3x^2.$$

Expressed in a more general form our rule then would be

D-1 $$\boxed{\frac{d}{dx}(x^n) = nx^{n-1}}$$

Now suppose that x^3 had had a greater coefficient than 1, a coefficient such as 2, for example.

What would be the derivative of $y = 2x^3$?

$$\frac{d}{dx}(2x^3) = y' = _____$$

Turn to page 43 to check your answer.

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ans. Multiplied the independent variable, x, by its exponent, 2, and decreased the exponent by one.

Yes, it's as simple as that: To "derive" the derivative of the function $y = x^2$, we have merely to multiply the x (actually, its coefficient, 1) by the exponent, 2, and subtract 1 from the exponent.

Thus,

$$y'(x^2) = 2 \cdot 1\ x^{2-1}, \text{ or } = 2x$$

Let's try this procedure with a slightly different function such as $y = x^3$.

What is the derivative of y in this case?

$y' =$ _____

Check your answer on page 41.

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ans. $y' = 3 \cdot 2x^{3-1} = 6x^2$

We have discussed what happens to the independent variable, x, when we find the derivative. But suppost there were a constant in the expression. What would ahppen to it?

For example, consider the function $y = 2 + x^3$.

We now know that the derivative of x^3 is $3x^2$. But what about the derivative of a constant, such as the 2 in this example?

The answer is that the derivative of a constant is zero, hence the 2 would just drop out. Without delving into the mathematical proof, the reason for this is that since a derivative represents a rate of change, and since a constant doesn't change, it simply drops out as a meaningless component of the derivative.

We express this symbolically as follows:

D-2 $$\boxed{\frac{d}{dx}(C) = 0}$$

Find the derivative of $y = 2x^4 + 7$.

$y' =$ _____

Turn to page 44 to check your answer.

BEST COPY AVAILABLE

ans. $y' = 4 \cdot 2x^{4-1} + 0$, or $y' = 8x^3$

What do you think would be the derivative of a function such as $y = x^2 + 2x + 1$?

Here we have an expression in which x appears in the first power as well as in a higher power. The procedure is, however, the same one we have been using to find the derivative of powers of x.

Here (from page 41) is the rule again:

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

With this rule in mind -- plus the rule regarding the derivative of a constant -- find the derivative of the expression $y = x^2 + 2x + 1$.

$y' =$ _____

Turn to page 46 to check your answer.

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In working the problems on pages 40, 41 and 42 we have been doing something of which you may not be aware. Although it follows logically from our prior discussion about finding the derivative of some power of x, it is time we examined it separately and recognized it as a rule in differentiating.

The rule is this: <u>When there is a constant as a multiplier, the constant remains a multiplier in the derivative</u>.

Thus, if $y = 4x^3$, then $\frac{dy}{dx} = 4 \cdot 3x^2 = 12x^2$ = the derivative.

Expressed symbolically this rule would appear as follows:

D-4 $$\boxed{\frac{d}{dx}(cv) = c\frac{dv}{dx}}$$

Do you realize you are solving problems in differential calculus? Perhaps not big ones, but finding the derivatives of functions of <u>any</u> kind is the heart of differential calculus.

Try this one just to prove to yourself you can do it:

What is the derivative of $y = 3x^3 + 2x^2 + 4x - 7$?

$y' =$ _____

Turn to page 47 to check your answer.

ans. $y' = 2x^{2-1} + 1 \cdot 2x^{1-1} + 0$
$= 2x + 2$

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This last problem also brought out another interesting point which I hope you inferred from your knowledge of algebra. This is the fact that the derivative of a variable with respect to itself is unity (one).

Thus, if $y = x$, then $y' = 1$. Why?

Following our rule for finding the derivative of a power of x (in this case the <u>first</u> power), $y'(x) = 1 \cdot x^{1-1}$. But according to the rules governing exponents, x^{1-1} is the same thing as $\frac{x^1}{x^1}$, and since <u>any</u> quantity divided by itself is unity, then it follows that $x^{1-1} = 1$.

Another, graphical, way of looking at this situation is that if $y = x$, then $dy = dx$, and the growth rates are equal, hence $\frac{dy}{dx} = \frac{dx}{dx} = 1$.

So add this useful piece of information to your increasing repertoire of knowledge about derivatives. Formally stated, therefore, our formula is

D-3 $$\boxed{\frac{d}{dx}(x), \text{ or } \frac{dx}{dx}, = 1.}$$

Continue on page 45.

BEST COPY AVAILABLE

ans. $y' = 3 \cdot 3x^{3-1} + 2 \cdot 2x^{2-1} + 1 \cdot 4x^{1-1} - 0$

$= 9x^2 + 4x + 4$

Now, what have we in fact discovered through the use of our empirically derived rules? This: We have found a simple, shortcut way to determining the derivative or instantaneous rate of change of a function!

To make sure you have these rules firmly in mind let's review them again briefly before going on.

	Rule	Meaning
D-1	$\frac{d}{dx}(x^n) = nx^{n-1}$	The derivative of x to the nth power is equal to n times x to the n-minus-1 power.
D-2	$\frac{d}{dx}(c) = 0$	The derivative of a constant is zero.
D-3	$\frac{d}{dx}(x) = 1$	The derivative of x (or of any variable) with respect to itself is one (1).
D-4	$\frac{d}{dx}(cv) = c\frac{dv}{dx}$	The constant remains a multiplier in the derivative.

To give yourself a little more confidence, try working the following problems, using the rules above.

Differentiate (that is, find the derivatives of) the following functions with respect to x:

1. $y = x^6$
2. $y = 3x^3$
3. $y = x^2 + x$
4. $y = 7 + 7x^2$
5. $y = 2x - 4 + 2x^5$
6. $y = 1 + 3x + x^9$
7. $y = 4x^2 - 4$
8. $y = 1$

Turn to page 48 to check your answers.

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ans. 1. $6x^5$ 2. $9x^2$ 3. $2x + 1$ 4. $14x$
5. $2 + 10x^4$ 6. $3 + 9x^8$ 7. $8x$ 8. 0

Did you get them all right? Actually, although we didn't treat it as such, successive differentiation of the terms of a polynomial (such as problems 3 through 7 on page 47) is generally considered to be governed by a separate rule. This rule simply says (in a common sense sort of way) that to differentiate a function that is made up of several terms connected by plus or minus signs, just differentiate one term after another in succession.

Written out symbolically the rule looks like this:

D-5 $$\boxed{\frac{d}{dx}(u + v + w + \ldots) = \frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx} + \ldots}$$

u, v, and w representing the various terms.

Now let's see if <u>you</u> can make up a rule.

Suppose we had a case in which the function was <u>divided</u> by a constant. How would you handle this?

See if you can differentiate the function shown below and then devise a rule to cover such a situation.

$$y = \frac{x^2}{2}$$

$$y' = _____$$

Rule:_____

_____

Check your answer on page 50.

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ans. 1. $\frac{3x}{2}$ 2. $10x^3$ 3. $7x^2$ 4. $\frac{4x}{5}$ 5. $\frac{12x^3}{7}$

There are many tricks to finding the derivatives of various kinds of functions and we are going to examine only a few of them. Just enough to get you started in the right direction and to give you some notion of what it is all about. The more difficult tricks -- and how to apply them to practical problems -- you will learn in your regular calculus course.

However, before getting back to The Mystery of the Thrown Ball, there is at least one more trick we should consider. This is the matter of how to differentiate a function having a fractional exponent.

Let us suppose, for example, we need to find the derivative of the function $y = \sqrt{x}$.

Do you recognize that x in this case has a fractional exponent? Remember: the square root of x can also be written as $x^{\frac{1}{2}}$, right? Therefore we can write the function as

$$y = x^{\frac{1}{2}}.$$

That being the case, then $y' =$ _____?

(Follow your regular rule for finding the derivative of some power of x.)

Turn to page 51 to check your answer.

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ans. Since $\frac{x^2}{2}$ is the same as $\frac{1}{2}x^2$, the $\frac{1}{2}$ is simply the constant coefficient of x^2; hence $y' = 2(\frac{1}{2})x^{2-1}$ or $y' = x$. Rule: To find the derivative of a function divided by a constant, treat the constant as a fractional multiplier.

Here is how we might write, symbolically, the rule derived on the preceding page:

D-6 $$\boxed{\frac{d}{dx}\left(\frac{y}{c}\right) = \frac{1}{c}\cdot\frac{dy}{dx}}$$

Another example of this rule would be as follows:

If $y = \frac{x^3}{9}$, then $y' = \frac{1}{9}\cdot 3x^2 = \frac{x^2}{3}$.

Or if $y = \frac{2x^3}{3}$, then $y' = \frac{2}{3}\cdot 3x^2 = 2x^2$.

The point is this: Treat the fraction as you would any other constant coefficient of the variable.

Perhaps working a few problems of this type will clarify things further and give you more confidence. Here they are.

Find the derivatives of the following functions:

1. $y = \frac{3x^2}{4}$ $\qquad y' =$ _____
2. $y = \frac{5x^4}{2}$ $\qquad y' =$ _____
3. $y = \frac{7x^3}{3}$ $\qquad y' =$ _____
4. $y = \frac{2x^2}{5}$ $\qquad y' =$ _____
5. $y = \frac{3x^4}{7}$ $\qquad y' =$ _____

Turn to page 49 to check your answers.

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ans. $y' = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2x^{\frac{1}{2}}} = \frac{1}{2\sqrt{x}}$

Did you remember your basic rule? Here it is again, a bit more simply stated: To differentiate a power (whether it is a fractional power or a whole number) multiply by the power and reduce the exponent by one.

You will see that we did exactly this in arriving at the answer shown on the previous page. And in case you may have forgotten what you learned in algebra about exponents, a negative exponent becomes positive when the term is moved from the numerator to the denominator of a fraction. This is because multiplying both numerator and denominator by the equivalent positive power has the effect of moving the variable with the negative exponent to the opposite side of the fraction bar and making it positive.

Try it for yourself, but be sure you have shown each step of the work before comparing your answer with that shown on page 52.

Convert $x^{-\frac{1}{2}}$ to a positive exponent.

Turn to page 52 to check your answer.

ans. $\frac{x^{\frac{1}{2}}}{x^{-}} \cdot \frac{x^{-\frac{1}{2}}}{1} = \frac{x^{0}}{x^{+}} = \frac{1}{x^{+}}$ (Any number to the zero power is 1, hence $x^{0} = 1$.)

Although there are a number of other derivative formulas (most of which you will encounter in your regular calculus course), we are only going to concern ourselves with two more of these -- the derivative of a product and the derivative of the quotient of two functions.

On page 48 we used the additional variables u, v and w to represent other functions of x. We are going to use two of these variables here, namely u and v, to simplify our explanation of how to differentiate a product. Because we have been using fairly simple expressions -- such as x^2, $3y^4$, $x^{\frac{1}{2}}$, etc. -- you may not see the need of introducing two more variables. But as you advance to working with more complex expressions you will come to appreciate the clarity that this little trick can bring to a problem.

For example, we said above we were going to talk about how to find the derivative of the <u>product</u> of two functions of x. Now suppose this product was $(x^2 + 3x + 4)(x^3 + x - 1)$. We would then have to write

$$\frac{d}{dx}(x^2 + 3x + 4)(x^3 + x - 1).$$

However, if we let $u = x^2 + 3x + 4$ and $v = x^3 + x - 1$, then we can simplify the whole thing and express the formula for the derivative of the product of two functions as

D-7 $$\boxed{\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}}$$

See if you can put this formula into words.

Turn to page 54 to check your wording.

ans. $(x^2 + 1)(1) + (x - 4)(2x) = 3x^2 - 8x + 1$

Having examined briefly the formula for the derivative of the product of two functions and the way in which this formula is used, let us take an equally brief look at the formula for the derivative of the quotient of two functions.

This formula is as follows:

D-8 $$\boxed{\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}}$$

Put into words we would say: The derivative of the quotient of two functions is equal to the denominator times the derivative of the numerator, minus the numerator times the derivative of the denominator, all divided by the denominator squared.

As usual, the procedure can be expressed more clearly in symbols than in words. (Remember: If you find yourself wondering why the product and quotient derivative formulas are the way they are, it would be good practice trying to derive them yourself using the delta method. Check your procedure with that given in any good calculus text in case you get lost along the way.)

Now let's see how the quotient formula works by differentiating the expression $y = \frac{2x^2 + 3}{x + 1}$.

Setting $u = 2x^2 + 3$ and $v = x - 1$, we get

$$\frac{dy}{dx} = \frac{(x - 1)(4x) - (2x^2 + 3)(1)}{(x - 1)^2} = \frac{4x^2 - 4x - 3}{(x - 1)^2}.$$

What would be the derivative of $y = \frac{x^3 - 5}{2x + 3}$?

Turn to page 55 to check your answer.

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ans. The derivative of the product of two functions is equal to the first times the derivative of the second, plus the second times the derivative of the first.

Very well, let's try using this formula. To do so we will find the derivative of the product of the two functions $x^2 + 3x + 4$ and $x^3 + x - 1$.

First, however, let us restate our formula so that it will be readily available to refer to:

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Let us also remind ourselves that, in this example,

$$u = x^2 + 3x + 4$$

and $$v = x^3 + x - 1.$$

Since $$\frac{du}{dx} = 2x + 3$$

and $$\frac{dv}{dx} = 3x^2 + 1,$$

then $$\frac{d}{dx}(uv) = \underbrace{(x^2+3x+4)}_{u}\underbrace{(3x^2+1)}_{\frac{dv}{dx}} + \underbrace{(x^3+x-1)}_{v}\underbrace{(2x+3)}_{\frac{du}{dx}}$$

Since multiplying these expressions and combining like terms where possible to simplify is purely an algebraic exercise and would add nothing to your understanding of the procedure, we will not take the problem any further -- although you are welcome to do so if you feel you need the practice. (Answer: $5x^4 - 12x^3 + 15x^2 + 4x + 1$.) At the moment we are only interested in demonstrating the procedure, or use of the formula, for finding the derivative of the product of two functions.

However, here is a somewhat simpler problem for you to complete by yourself.

$$\frac{d}{dx}(x^2 + 1)(x - 4) = _____$$

Check your answer by turning to page 53.

BEST COPY AVAILABLE

ans. $y' = \dfrac{(2x+3)(3x^2) - (x^3-5)(2)}{(2x+3)^2} = \dfrac{4x^3 + 9x^2 + 10}{(2x+3)^2}$

On page 47 we summarized the rules for differentiation which we had covered to that point. Now let us summarize the additional rules we have worked with since then.

	Rule	Meaning
D-5	$\frac{d}{dx}(u+v+w+\ldots) = \frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx} + \ldots$	To differentiate the sum of several terms connected by plus or minus signs, differentiate each term in succession.
D-6	$\frac{d}{dx}(\frac{v}{c}) = \frac{1}{c} \cdot \frac{dv}{dx}$	When differentiating a function divided by a constant, treat the constant as a fractional multiplier.
D-7	$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$	The derivative of the product of two functions is equal to the first times the derivative of the second, plus the second times the derivative of the first.
D-8	$\frac{d}{dx}(\frac{u}{v}) = \frac{v(\frac{du}{dx}) - u(\frac{dv}{dx})}{v^2}$	The derivative of the quotient of two functions is equal to the denominator times the derivative of the numerator, minus the numerator times the derivative of the denominator, all divided by the square of the denominator.

Turn the page now and we'll look a little further into the meaning and application of differentiation.

Now that you have had a little experience in differentiating functions to find their instantaneous rates of change (and that is what we have been doing, in case you have forgotten), let us see how the procedure works in the case of the ball, rising and falling under the influence of gravity.

Our equation for the height of the ball was, if you will recall, $h = 128t - 16t^2$. From what we have learned, therefore, we can now differentiate this expression to find the time rate of change of height. Thus,

$$\frac{dh}{dt} \text{ or } h' = 1 \cdot 128t^{1-1} - 2 \cdot 16t^{2-1}$$

$$\text{or} \quad h' = 128 - 32t.$$

That is, the rate of change of height (distance) with time is equal to 128 - 32t. Hence for t = 2 seconds (the instant we originally selected for analysis), h', or v (for velocity), = 128 - 64, or 64 feet per second, precisely the instantaneous velocity value for t = 2 which we found by approximation on page 20.

What would be the value of h' for t = 4?

h' = _____

Turn to page 58 to check your answer.

Mathematical symbols are distressing only if you don't understand them. The ones we will use in our general approach to finding a derivative are all ones with which you are acquainted:

Δx = a little bit of x

Δy = a little bit of y

$f(x)$ = any function of x

So this time instead of using the <u>specific</u> function $y = x^2$, let us substitute for x^2 the more general expression $f(x)$. This gives us

$$y = f(x).$$

Now considering, as we did on page 24, a point Q on the curve $f(x)$, a short distance away (Δx horizontally and Δy vertically) from the point P, its coordinates will be $y + \Delta y$ and $x + \Delta x$.

Substituting these coordinates of the point Q for y and x in our equation $y = f(x)$ therefore gives us:

Turn to page 59 to check your answer.

ans. $128 - 32 \cdot 4 = 128 - 128 = 0$

Does that answer surprise you?

It won't if you will turn back to page 13 and observe from the graph that the ball was rising (its speed decreasing) between $t = 0$ and $t = 4$. After $t = 4$ the ball was falling (its speed increasing). At $t = 4$, therefore, the instantaneous velocity was zero; the ball was momentarily at rest, neither rising nor falling.

In pages 28 to 33 we illustrated, both algebraically and graphically, what is generally referred to as the delta (Δ) approach to finding a derivative. For this purpose we used the specific function $y = x^2$.

Now, in order to keep at least a minimum of faith with the professional mathematicians -- and without, I think, alarming you unduly -- we will do the same thing using a general case.

Turn to page 57, please, and we'll have a go at it.

ans. $y + \Delta y = f(x + \Delta x)$

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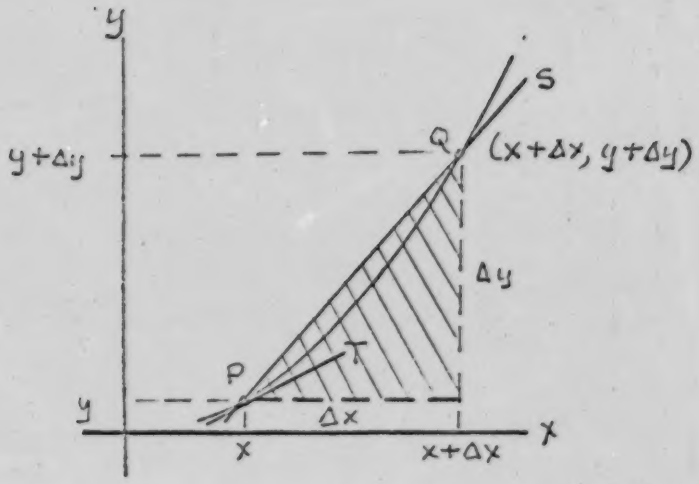

Knowing, then, that the value of the function $y = f(x)$ at the point Q is $y + \Delta y = f(x + \Delta x)$, and recalling from the general statement of the function that $y = f(x)$, we can substitute this last value for y in the equation

$$y + \Delta y = f(x + \Delta x)$$

giving us $$f(x) + \Delta y = f(x + \Delta x)$$

from which $$\Delta y = f(x + \Delta x) - f(x)$$ (subtracting $f(x)$ from both sides).

Dividing both sides by Δx we get

$$\boxed{\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}}$$

What does this last, boxed-in, expression represent? See if you can put it into words in the space provided below.

_____

_____

Turn to page 60 to check your answer.

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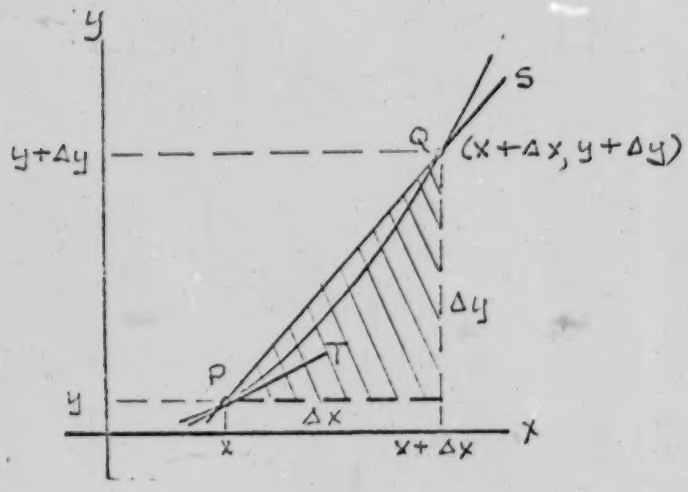
ans. The general expression for the slope of the secant line S, or the average rate of change (growth rate) of the function with respect to x.



Compare this expression, $\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$, for the secant S with that for the function $y = x^2$ found on page 29. The expression $\frac{\Delta y}{\Delta x} = 2x + \Delta x$ on page 31 represented the average rate of change in the <u>specific</u> function $y = x^2$ over the interval Δx. On the other hand $\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$ represents the average change of <u>any</u> function (represented by $y = f(x)$) over the interval Δx.

To find the <u>general expression</u> for the derivative, then, we again imagine Q to approach nearer and nearer the point P. That is, we allow Δx to approach zero as a limit. This gives us

$$\lim_{x \to 0} \frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{dy}{dx}.$$

This, then, is the general expression for the derivative of a function, or the <u>instantaneous rate of change</u> of y with respect to x (or whatever "name" we give to the variables -- h and t, x and y, u and v, etc.).

CHAPTER 5: DIFFERENTIAL CALCULUS PUT TO WORK

No doubt you are beginning to wonder if all this is leading to something useful.

As pointed out earlier there are many, many practical applications for differential calculus, a great number of which you will be exposed to in your regular calculus course. Since this book was designed simply to initiate you into some of the fundamental concepts at a slower pace than usually is possible in a standard academic course, we will not attempt to explore any more types of standard derivatives. Nor will we get involved, to any real extent, with applications. However, it would not be fair to leave you without having had the fun of applying some of the fundamental things you have learned.

Therefore we will look at some examples together. First, however, a word of caution.

We have used the variables x and y most frequently because that are considered general variables (that is, they represent any variables) and because they also represent the familiar coordinates of the Cartesian coordinate system. But don't get the idea that they are the only two variables used in calculus! We have already had an example of two other, specific, variables in the problem of the thrown ball.

What were these two variables? _____

Turn to page 62 to check your answer.

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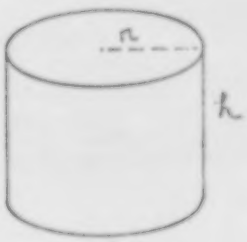
ans. t (time) and h (height)

Example 1

Suppose we wished to know the rate of variation of the volume of a cylinder with respect to its radius when the radius is 5 inches and the height of the cylinder is 20 inches. In other words, how much will the volume (in cubic inches, of course) change for a change of one inch in the radius of the cylinder under the particular conditions where r = 5 inches and h = 20 inches?



This situation is shown in the sketch at the right.

We must start with the formula for the volume of a cylinder, namely, $V = \pi r^2 h$.

And since our problem is to find the rate of change of V with respect to r, we take the derivative of V with respect to r, namely,

$$V = \pi r^2 h$$

$$V' = 2\pi rh$$ (h, of course, is a constant in this problem)

Substituting the given values r = 5 and h = 20 we get

$$V' = 2 \cdot \pi \cdot 5 \cdot 20 = 628$$ cubic inches/inch change of radius.

What this means is that at the particular point where r = 5 inches the volume of the cylinder is changing at the rate of 628 cubic inches for a change of one inch in the radius.

Although you may feel that this is not the kind of problem you are apt to encounter in your kitchen or workshop, it is a simple example of a very common type of problem found in engineering. How might it be of use, for example, in designing containers?

Turn to page 64 to check your answer.

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ans. $D' = 2t + 6t^2 = 20 + 600 = 620$ feet/second

Example 3

Acceleration is defined as the rate of change of velocity with respect to time. If the velocity (in feet per second) of a certain airplane in a dive is $v = 300 + 4t^2$, where t is the number of seconds since the dive began, what is the formula for the acceleration and what is the acceleration value for $t = 10$?

Since acceleration is the rate of change of velocity with time, if we take the derivative of the velocity formula given above it should yield the formula for acceleration. Right? Let's try it.

$$v = 300 + 4t^2$$

Taking the derivative of v with respect to t, $v' = 8t$ = formula for acceleration.

But since v' represents acceleration, we can write it as

$$a = 8t.$$

Therefore, for $t = 10$, $a = 80$ feet/second^2.

Question: What would be the acceleration formula if the velocity was $v = 50t + t^3$, and what would be the acceleration value when $t = 4$ seconds?

Turn to page 65 to check your answer.

ans. For a certain height of can (circular) container one could estimate added volume for increased radius.

Your answer may differ somewhat from that given, but the important thing is to understand what the derivative of a function such as this means and how it can be used in a practical way.

Example 2

An object moving in a straight line is $t + t^3$ feet from its starting point after t seconds. What is its velocity after 10 seconds?

If the object is $t + t^3$ feet from its starting point after 10 seconds, then $t + t^3$ must represent the distance it has travelled. We can write this as

$$D = t + t^3$$

and we have our function, or relationship, between distance and time. And since velocity is simply the rate of change of distance with time, if we take the derivative of D with respect to t we should have an expression for velocity. Thus,

$$D = t + t^3$$

and

$$D' = 1 + 3t^2.$$

Substituting the value 10 (seconds) for t gives us

$D' = 301$ feet/second, the instantaneous velocity of the object at $t = 10$.

Question: What would the velocity be after 10 seconds if the object were moving at the rate $D = 1 + t^2 + 2t^3$?

Turn to page 63 to check your answer.

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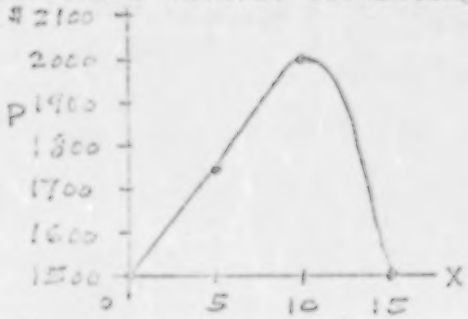
ans. $a = 50 + 3t^2$; for $t = 4$, $a = 50 + 48 = 98$ feet/sec^2

Example 4

A certain firm makes a profit of $P each month when it produces x tons of a certain commodity, where $P = 1500 + 15x^2 - x^3$. What is the most advantageous monthly output for the company?

Obviously the most advantageous output will be the one that produces the greatest profit in dollars. The question is, then, what production tonnage (x) will yield the most dollars? This means we are seeking the maximum rate of change P with respect to x.

Let's start out by solving this graphically, plotting the graph of P as a function of x and finding out where the maximum (high) point of the curve is. To do this we will use the values x = 0, 5, 10, and 15 and sketch in the rest of the curve without tabulating values.



$P	X
1500	0
1750	5
2000	10
1500	15

The tonnage that yields a maximum profit is, x = _____, for which P = _____.

The rate of change of P with respect to x is (from the formula at the top of the page), P' = _____.

What would you expect the value of P' to be at the point x = 10?

Turn to page 66 to check your answers

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ans. x = 10, P = $2,000; $P' = 30x - 3x^2$; zero.

The first answer simply represents the coordinates of the high point of the curve; the table of values, of course, gives the same information.

The second answer represents the derivative of P with respect to x in the given formula, that is, the instantaneous rate of change of profit dollars with respect to tons produced.

If you got the answer to the third question correct you did very well indeed, for this involved a little thinking about what the derivative means graphically. Do you recall we said earlier that it represents the tangent to the curve at a point, or the slope of the curve? It isn't hard to visualize the slope of a curve that is headed either up or down. But what does the slope look like when the curve is going neither up or down? This is the situation we find at the high point of the curve.

If you got the correct answer you realized that the slope would be zero at the maximum point (or at a minimum point, though we won't go into that here), because the tangent would be parallel to the x-axis.

If all this is so, then substituting the value 10 for x in the derivative of P, namely $P' = 30x - 3x^2$, should produce a value of zero for P'. Try it.

For x = 10, P' = _____ = _____.

Turn to page 68 to check your answer.

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ans. Trigonometric, Inverse-Trigonometric, Exponential, Logarithmic

Don't become alarmed. We are not going to become involved in all the possibilities that exist for differentiating the various types of transcendental functions. We are really only going to look at two: the derivative of a trigonometric function and of a logarithmic function.

However, since it won't help you to know how to differentiate such functions unless you first can recognize them, look at the functions below and see if you can identify them as either algebraic, trigonometric or exponential.

1. $y = \sin(x^2 - 3)$ _____

2. $y = e^x$ _____

3. $y = 2x^3 - 3$ _____

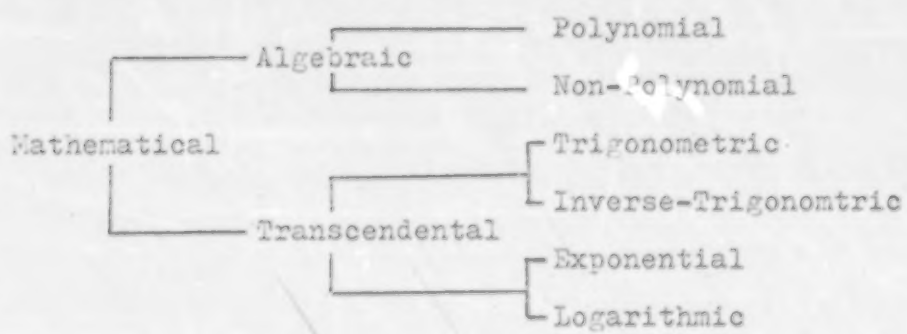
4. $y = 2 \cos x \sin x$ _____

Turn to page 69 to check your answer.

ans. $P' = 300 - 3 \cdot 100 = 300 - 300 = 0$

Since the possibilities for application are nearly limitless we will not include any further examples. You will find all of these you want in any first-year book on calculus. Remember: our purpose here is not to supplant such a text, merely to supplement it by introducing you to the basic concepts at a leisurely pace.

Before leaving this brief excursion into differential calculus, however, there is another class of functions with which you should at least be familiar. These are known as "transcendental" functions. And in case you don't recognize this term from your study of algebra, a transcendental function is one which is non-algebraic, although it is an important branch of the family of mathematics. Below is a chart that should help to refresh your memory regarding the various branches of the family of functions and equations.



Write below the names of the four kinds of transcendental functions shown in the chart above.

1. _____ 2. _____ 3. _____ 4. _____

Turn to page 67 to check your answer.

ans. 1. trigonometric 3. algebraic
2. exponential 4. trigonometric

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Example 2 is an exponential function; example 3 is an algebraic (power) function. Be careful that you don't confuse them. An exponential function (which is a kind of transcendental function) consists of a constant (or variable) with a variable exponent. A power function (which is a kind of algebraic function) consists of a variable with a constant exponent.

I'm sure you had no difficulty recognizing examples 1 and 4 as trigonometric functions, and this is the type of function we will examine first.

The derivative of any of the trigonometric functions can be arrived at by means of the delta process, however we are not going to make you wade through this. We are only going to consider the sine function and will simply state that

D-9 $$\boxed{\frac{d}{dx}(\sin v) = \cos v \frac{dv}{dx}},$$

where v is simply some function of x.

Applying this rule to example 1 from page 70 we get

$y = \sin(x^2 - 3)$, hence $v = x^2 - 3$ and $\frac{dv}{dx} = 2x$

therefore $y' = 2x \cos (x^2 - 3)$.

Use this approach to find the derivative of the following function:

$y = \sin(x^3 - 4x)$; $y' =$ _____

Turn to page 70 to check your answer.

ans. $y' = 2(x^2 - 2)\cos(x^3 - 4x)$
$= 2x^2 - 4\cos(x^3 - 4x)$

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Your textbook (or any table of standard derivatives) will give you the formulas for finding the derivatives of the other trigonometric functions. To use them you just apply them in the same way we did on the previous page for the sine function. However, let us turn our attention now to the problem of finding the derivative of a logarithmic function. This is interesting because it introduces the concept of the so-called "natural" logarithmic base e.

In algebra we are used to working with the logarithmic base 10, the base used for "common" or Briggsian logs. Let us consider, however, what the derivative might be of the function $y = \log_b x$ if we treat it temporarily as an algebraic expression and don't worry about what the base b represents.

Using the delta method we arrive at the expression

$$\frac{\Delta y}{\Delta x} = \frac{1}{x}\log\left(1 + \frac{\Delta x}{x}\right)^{\frac{1}{\frac{\Delta x}{x}}}$$

This looks a bit messy, but as we allow Δx to approach 0, the exponential function $\left(1 + \frac{\Delta x}{x}\right)^{\frac{1}{\frac{\Delta x}{x}}}$ appears to approach the value 2.718... as a limit, hence the derivative becomes

$$\frac{dy}{dx} = \frac{1}{x}\log_b 2.718...$$

Now comes the trick. If we allow b = e = 2.718..., then the derivative simplifies very beautifully to

$$\frac{d}{dx}(\log_e x) = \frac{1}{x}$$

Continue on page 71.

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Thus we arrive at the base e = 2.718... for what is termed the "natural" logarithmic base, or the base of natural logs. The only thing natural about it is, of course, that it is "naturally convenient" in order to make the standard, step-by-step process of differentiation work out simply for a logarithmic function.

Logarithms taken to the base e also are known as Naperian logs, after the man who first calculated the tables for them. Because of their convenience, Naperian or natural logs are used almost exclusively in calculus and advanced mathematics.

Finally, then, the formula for the derivative of a logarithmic function (v) of x is

D-10 $$\boxed{\frac{d}{dx}(\log_e v) = \frac{1}{v}\frac{dv}{dx}}$$

Let us take as an example of the use of this formula the function $y = \log(x^2 - 2)$. (We will not continue to write in the base e but henceforth will consider it understood.)

Since v in this case is $(x^2 - 2)$, our formula tells us to take the derivative of this expression with respect to x and to place this over the function itself. Therefore,

$$y' = \frac{2x}{x^2 - 2}.$$

Now suppose you try differentiating the function

$$y = \log(x^2 + 3x - 2).$$

Turn to page 72 to check your answer.

from page 71

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ans. $y' = \dfrac{2x + 3}{x^2 + 3x - 2}$

So now we have arrived at the derivative formulas for two transcendental functions -- one trigonometric and one logarithmic -- and the last two we will investigate. They are:

	Rule	Meaning
D-9	$\frac{d}{dx}(\sin v) = \cos v \frac{dv}{dx}$	The derivative of the sine of v (some function of x) equals the cosine of the function times the derivative of the function with respect to x.
D-10	$\frac{d}{dx}(\log_e v) = \frac{1}{v}\frac{dv}{dx}$	The derivative of the logarithm (to the base e) of some function of x is equal to the derivative of the function with respect to x times 1 over the function itself.

Although there are many other standard derivative formulas or rules, they can all be arrived at by the same general delta process we have used thus far.

In brief, the approach we have followed in the foregoing pages is all there is to differential calculus. Everything else is just a systemmatic application of the same basic idea to different types of functions for different special purposes. Once we have used the delta process to find the general rules, we then use these general rules (formulas) to solve problems because they make the solution faster and simpler.

With this in mind turn to the next page where you will find a quick review of everything we have covered on the subject of differential calculus, after which we will proceed to explore the companion subject of integral calculus.

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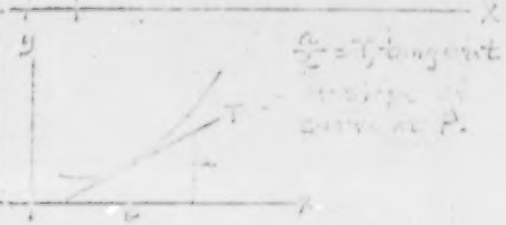
Review of differential calculus concepts.

Review Item	Page ref.	Example
1. In mathematics when we speak of "approaching a limit" the limit referred to usually is zero.	8	When one is running out of gas, the amount remaining in the tank is approaching zero as a limit.
2. The notion of approaching zero as a limit can be represented symbolically.	10	If G_r represents the gas remaining (in the example above) then we can write this as $G_r \longrightarrow 0$
3. A major problem that led to the development of calculus was that of how to determine the instantaneous, rather than the average, velocity of an object whose speed varies with time.	11	A free-falling body, such as an object thrown into the air or dropped from a height, is an example of one whose velocity varies with time.
4. The speed, at any given instant, of a free-falling body can be estimated quite closely by calculating its average speed over shorter and shorter intervals.	14, 16	(see pages 14 and 16)
5. As an aid to understanding the relationship between two variables (such as height or distance with time, x and y, etc.) it is often helpful to draw a graph of that relationship.	21	Relationships such as: $h = 128t - 16t^2$, or $y = x^2$
6. The shape of the resulting curve represents the rate at which either variable is changing value with respect to the other variable.	24	
7. The slope of a curve at any given point (which we measure by means of a line tangent to the curve at that point) represents the direction of the curve.	24	

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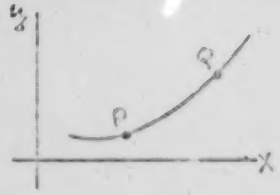
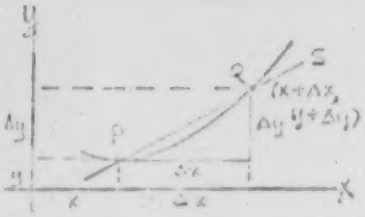
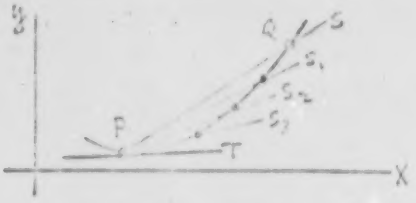
Review Item	Page ref.	Example
8. To assist in analyzing the relationship between the two variables in a function such as $y = x^2$, we graph the function, select a fixed point, P, and another point, Q, on the curve a short distance from P.	25	
9. A straight line passing through the two points is called the "secant" (S), and the right triangle formed thereunder has as its sides the two coordinate distances (Δx and Δy) between the points.	25	
10. The slope of the secant, S, is then Δy over Δx.	26	Slope $S = \frac{\Delta y}{\Delta x}$
11. In order to observe how the relationship between x and y varies along the curve, we imagine the point Q to gradually approach the point P, and hence the secant S to gradually approach the tangent line T.	31	
12. To find a mathematical expression for the slope of the secant line as Q approaches P, we substitute the coordinates of Q ($x + \Delta x$ and $y + \Delta y$) in the equation of the curve, namely, $y = x^2$. This gives us the expression $\frac{\Delta y}{\Delta x} = 2x + \Delta x$ for the slope of S.	30, 31	Coordinates of Q = $(x + \Delta x)$ and $(y + \Delta y)$. Substituting these values for x and y in the equation $y = x^2$ gives us: $y + \Delta y = x^2 + 2x \cdot \Delta x + \overline{\Delta x}^2$ or, substituting x^2 for y and dividing both sides by x, $\frac{\Delta y}{\Delta x} = 2x + \Delta x$

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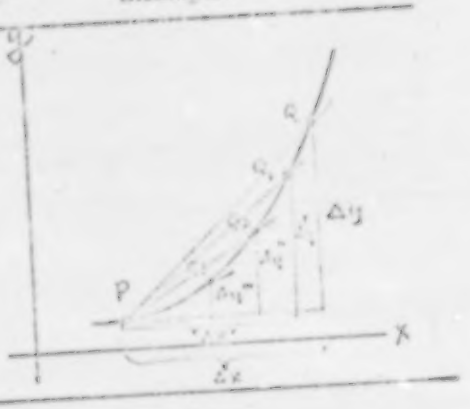
Review Item	Page ref.	Example
13. As Q approaches P, Δx gets shorter and shorter, that is, it approaches zero as a limit.	31	
14. When Q arrives at the point P, Δx does become zero, thus dropping out of the expression $\frac{\Delta y}{\Delta x} = 2x + \Delta x$, and the secant line becomes (coincides with) the tangent to the curve at P. Theoretically	33	
15. What occurs in item 14 above can be expressed symbolically as shown in the example opposite; verbally we say that "the limit of $\frac{\Delta y}{\Delta x}$, as Δx approaches zero as a limit, is 2x."	33	$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 2x$
16. The name given to the limiting value of $\frac{\Delta y}{\Delta x}$ as Δx approaches zero is "derivative," and it is written as $\frac{dy}{dx}$. A derivative, then, is a limiting value.	38	$\lim_{x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$ = derivative $\frac{dy}{dx}$ means the derivative of y with respect to x.
17. The derivative of y with respect to x in the expression $y = x^2$ is 2x.	38	$\frac{dy}{dx}(x^2) = 2x$
18. There are four common ways of expressing the derivative of y with respect to x.	40	$\frac{dy}{dx} = y' = f'(x) = \frac{d}{dx}y = \frac{d}{dx}f(x)$
19. Looking at item 17, it is apparent that we could have found the derivative of $y = x^2$ simply by multiplying the right-hand member of the equation by the exponent, 2, and then reducing the original exponent by one to get the new exponent.	42	$y'(x^2) = 2 \cdot x^{2-1}$ $= 2x$ or $y' = 2x$

Review Item	Page ref.	Example
20. The general rule for finding the derivative of some power of x is: $\frac{d}{dx}(x^n) = n \cdot x^{n-1}$ We read this as: The derivative of x to the nth power is equal to n times x to the n-minus-1 power.	44	$\frac{d}{dx}(x^4) = 4x^3$
21. The derivative of a constant is zero. Thus, $\frac{d}{dx}(c) = 0$	43	$\frac{d}{dx}(8) = 0$
22. The derivative of x (or of any variable) with respect to itself is one (1).	46	$\frac{d}{dx}(x) = 1$
23. The term "differentiate" means "find the derivative of."	47	To differentiate the function $y = x^5$ means to find the derivative of y with respect to x in the function $y = x^5$.
24. To differentiate a function composed of several terms connected by plus or minus signs, simply differentiate one term at a time. Symbolically expressed: $\frac{d}{dx}(u + v + w + \ldots) = \frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx} + \ldots$	48	If $y = 2x^4 + 3x^2 - 4x$, then $y' = 8x^3 + 6x - 4$.
25. To find the derivative of a function divided by a constant, treat the constant as a fractional multiplier. $\frac{d}{dx}(\frac{y}{c}) = \frac{1}{c} \cdot \frac{dy}{dx}$	50	If $y = \frac{3x^2}{4}$ then $y = \frac{3}{4}x^2$ and $y' = \frac{3}{2}x$ or $\frac{3x}{2}$

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Review Item	Page ref.	Example
26. To differentiate a fractional power, multiply by the fraction and reduce the exponent by one (just as you would with any exponent).	51	$y = x^{\frac{1}{2}}$ $y' = \frac{1}{2}x^{-\frac{1}{2}}$
27. The derivative of the product of two functions is equal to the first times the derivative of the second, plus the second times the derivative of the first. Thus. $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$	52	$\frac{d}{dx}(x^2)(x^3)$ $= x^2(3x^2) + x^3(2x)$ $= 3x^4 + 2x^4$ $= 5x^4$
28. The derivative of the quotient of two functions is equal to the denominator times the derivative of the numerator, minus the numerator times the derivative of the denominator, all divided by the denominator squared. $\frac{d}{dx}(\frac{u}{v}) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$	53	$\frac{d}{dx}(\frac{x^5}{x^2}) =$ $\frac{x^2(5x^4) - x^5(2x)}{x^4}$ $= \frac{5x^6 - 2x^6}{x^4}$ $= \frac{3x^6}{x^4} = 3x^2$
29. Taking the derivative of height (h) with respect to time (t) in the equation giving the change in height (with time) of the ball thrown into the air, yields the formula for the <u>instantaneous velocity</u> of the ball at any given instant.	56	$h = 128t - 16t^2$ $h' = 128 - 32t$ When t = 2, then h', or v, = 64 ft/sec. (note that h' actually is v, the velocity or time rate of change of the position of the ball)
30. The <u>general expression</u> for the derivative is obtained in the same general way that we found the derivative of the <u>specific</u> function $y = x^2$.	57, 59, 60	If $y = f(x)$, then $\lim_{x \to 0} \frac{\Delta y}{\Delta x}$ $= \frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{dy}{dx}$
31. <u>Transcendental</u> functions are non-algebraic functions.	68	Trigonometric, exponential and logarithmic functions.

Review Item	Page ref.	Example
32. The derivative of the sine of v (some function of x) equals the cosine of the function times the derivative of the function with respect to x. Thus, $\frac{d}{dx}(\sin v) = \cos v \frac{dv}{dx}$	69	$y = \sin x^2$ $y' = 2x \cos x^2$
33. In calculus and advanced mathematics the so-called "natural base," e, (equal to 2.718...) is used as the base for logarithms.	71	(See page 70)
34. The derivative of the logarithm (to the base e) of some function of x is equal to the derivative of the function with respect to x, multiplied by 1 over (that is, the reciprocal of) the function itself. Thus, $\frac{d}{dx}(\log_e v) = \frac{1}{v} \frac{dv}{dx}$	71	$y = \log (x^3 + 3)$ $y' = \frac{3x^2}{x^3 + 3}$

And now since you probably would like to discover how much you have learned, turn to page 79 and take the short self-quiz you will find there.

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Self-Quiz on the basic concepts of Differential Calculus

(Circle the correct answer or fill in the missing information)

1. Differentiate the expression $y = 3x^4$. ans. _____

2. Find the derivative of s with respect to r in the equation $s = 8r^3$. ans. _____

3. The symbol Δ is used in calculus to mean _____

4. The derivative (rate of change) of distance with respect to time is called velocity. True False

5. A long-standing mathematical problem whose solution led to the invention of differential calculus was that of finding a way to determine the _____ velocity of an object. (average / instantaneous)

6. The derivative of a constant is _____.

7. $\frac{\Delta y}{\Delta x}$ and $\frac{dy}{dx}$ mean the same thing. True False

8. The symbol $\longrightarrow$ means _____.

9. Find the derivative of y with respect to x in the expression $y = 5x^{\frac{3}{2}}$. ans. _____

10. In the figure at the right what is the line passing through the two points of the curve called? ans. _____

11. Differentiate the expression $y = x^3+2x^2-4x+1$. ans. _____

12. The derivative of any variable with respect to itself is _____.

Self-Quiz (continued)

13. The slope of a curve at a point is measured by the angle between the_____ to the curve at that point and the x-axis.

14. Find the derivative of y with respect to x in the expression $y = x^2$ using the delta (Δ) method. Show all steps.

15. The formula for the area of a circle is (as you probably recall) $A = \pi r^2$. Find the expression for the <u>rate</u> at which the area is changing with respect to the radius, and evaluate this expression for $r = 2$ feet.

ans.(1)_____

ans.(2)_____

16. Write the expression for $\frac{dy}{dx}$ in terms of Δy and Δx.

ans._____

17. An automobile moving in a straight line is a distance of $2t + 3t^2$ from its starting point after t seconds. What is its velocity after 12 seconds. (Remember: The derivative of distance with respect to time <u>is</u> velocity.)

ans._____

18. If the velocity of an object is given by the formula $v = 200 + 5t^2$, what would be its <u>acceleration</u> value for $t = 8$? (The velocity is in feet per second. Remember that acceleration is the time rate of change of velocity.)

ans._____

Self-quiz (continued)

19. Evaluate the <u>slope</u> of the curve defined by the expression $y = x^2 - 4x$ for the value $x = 2$. ans._____

20. In the previous problem what does the <u>value</u> of the slope tell us about the curve at the point $x = 2$? ans._____

21. Find the derivative of y with respect to x in the following expression: $y = (2x+1)(x^2-2)$. $y' =$ _____ (Use the product formula from page 52.)

22. Using the quotient formula from page 54 differentiate the expression $y = \frac{x^2 + 1}{x^2 - 1}$. ans._____

23. The expression $y = \tan(x^2 + 3)$ is a transcendental function. True False

24. What is the derivative of the expression $y = \sin(x^3 + 1)$? ans._____

25. Find y' if $y = \log(x^3 + 1)$. ans. $y' =$_____

Turn to page 82 to check your answers.

Answers to Self-Quiz on Differential Calculus

		Page ref.
1.	$12x^3$	41
2.	$s' = 24r^2$	41
3.	a little bit of, or an increment of; thus, Δx means a little bit of x.	25
4.	True	56, 64
5.	instantaneous	2
6.	zero	43
7.	False. $\frac{\Delta y}{\Delta x}$ represents the average rate of growth of a function, or average rate of change. $\frac{dy}{dx}$ is the limiting value of the ratio $\frac{\Delta y}{\Delta x}$ as the interval Δx of the independent variable approaches the limit zero. Therefore, $\frac{dy}{dx}$ is the instantaneous rate of change of a function at the point in question.	38
8.	approaches	7
9.	$y' = \frac{5x^{-\frac{1}{2}}}{2}$	52
10.	the secant	23
11.	$y' = 3x^2 + 4x - 4$	47,48
12.	one (1)	46
13.	tangent	24
14.	(see page 38)	37,38
15.	$A' = 2\pi r$; when $r = 2$, $A' = 4\pi$, or the instantaneous rate of change of the area of a circle when r = 2 feet is approximately 12.56 square feet per unit change in the radius.	62

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Answers to Self-Quiz (continued)

Page ref.

16. $\lim_{x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$ — 38

17. $d = 2t + 3t^2$ — 64

$d' = v = 2 + 6t$

when $t = 12$ seconds, $v = 2 + 6(12) = 74$ ft/sec

18. $v = 200 + 5t^2$ — 63

$v' = a = 10t$

when $t = 8$ ft/sec, $a = 80$ ft/sec/sec

19. $y = x^2 - 4x$ — 66

$y' = \text{slope} = 2x - 4$

when $x = 2$, $y' = 0$

20. Since the slope of the curve = 0, the tangent to the curve at the point $x = 2$ is parallel to the x-axis, meaning that this is either a maximum or a minimum point on the curve and that the curve is changing direction. — 66

21. $y = (2x + 1)(x^2 - 2)$ — 54

$y' = (2x + 1)2x + (x^2 - 2)2$

$= 4x^2 + 2x + 2x^2 - 4$

$= 6x^2 + 2x - 4$

22. $y = \frac{x^2 + 1}{x^2 - 1}$ — 53

$y' = \frac{(x^2 - 1)2x - (x^2 + 1)2x}{(x^2 - 1)^2}$

$= \frac{-4x}{(x^2 - 1)^2}$

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Answers to Self-Quiz (continued)

	Page ref.
23. True	68
24. From the formula on page 72, $y = \sin(x^3 + 1)$ since $v = x^3 + 1$, then $\frac{dv}{dx} = 3x^2$ and $y' = 3x^2\cos(x^3 + 1)$	69
25. Again from page 72, using the log formula for the derivative, $y = \log(x^3 + 1)$ since $v = x^3 + 1$ and $\frac{dv}{dx} = 3x^2$, then $y' = \frac{3x^2}{x^3 + 1}$	71

If you got 20 or more right you did very well indeed. If you missed more than five you would do well to review the items you missed.

Now it's time for us to consider the subject of integral calculus, the counterpart of differentail calculus. So please turn to page 85 and we'll proceed.

CHAPTER 6: A LOOK AT INTEGRAL CALCULUS

In arithmetic and algebra we have several operations that are the *inverse* of one another. That is, one operation *undoes* the other.

For example, subtraction is the inverse of addition because it undoes addition. Division is the inverse of multiplication because it undoes multiplication. Similarly, taking the square root of a number is the inverse of squaring the number (except, of course, that in taking square root we wind up with two answers since the original number could have been either positive or negative).

The relationship between differential calculus and integral calculus is quite similar, for integral calculus is, in effect, the *inverse* of differential calculus. For example, the process of differentiating the expression $y = x^3$ consists of finding the derivative of y with respect to x. From what we have learned in the preceding chapters we know that this would be $y' = 3x^2$.

Suppose, however, we were *given* the expression $y' = 3x^2$ and asked to perform the process of *integration* (not in its social sense, please) on it. This would mean finding the *original* expression from which $y' = 3x^2$ was derived. What would we have to do to $3x^2$ to turn it into x^3?

Turn to page 86 to check your answer.

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ans. Divide by 3 and increase the exponent by one.

Let's apply this procedure and see if it works.

$$y' = 3x^2$$

Dividing this expression by 3 and increasing the exponent by one gives us

$$y = \frac{3x^{2+1}}{3} = x^3,$$ our original expression.

Seems to work, doesn't it? This is integration.

Now there is a rather odd-looking symbol, known as an integral sign, that is used to indicate this process of integration and it is this: $\int$. Basically it is simply an elongated S. Thus if we wished to indicate that the process of integration was required we would place this sign in front of the expression. Apply it to the problem above we would get: $y = \int 3x^2 = x^3$.

However, there is still one thing lacking, namely, some indication as to the variable with respect to which the integration is to be performed. In the above example since we wish the integration to be performed with respect to the variable x, we indicate this by adding dx after the term $3x^2$ (which, by the way, is called the integrand). This gives us $\int 3x^2 dx = x^3$.

In general terms then: To integrate a simple exponential function such as the one we have used here, increase the exponent by one and divide the expression by the new exponent. Try this on the following problems:

1. $\int 5x^4 dx =$ _____
2. $\int 4x^3 dx =$ _____
3. $\int 6x^2 dx =$ _____
4. $\int x dx =$ _____

Turn to page 87 to check your answers.

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ans. 1. x^5 2. x^4 3. $2x^3$ 4. $\frac{1}{2}x^2$

For simplicity's sake we have so far omitted something that actually is quite important. I wonder if you know what it is? Here's a hint.

Sometimes in "going backwards" by inverse operation to find an original expression we run into in inherent uncertainty. For instance, in taking the square root of a number we can't be sure whether the original term was positive or negative. Thus, the square root of 4 could be either +2 or -2. How then do we know which is correct? The answer is, we don't. We simply have two possible answers, both of which may be correct.

Bearing this in mind and the fact that in the process of differentiation constants drop out, what do you think is wrong with accepting x^3 as the complete answer to $\int 3x^2 dx$?

Turn to page 88 to check your answer.

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ans. There may have been a constant in the original expression which does not appear in our answer.

Of course. If we start out to perform the process of integration on an expression which already is a derivative -- and this is the only kind of expression we can integrate -- then we have no way of knowing what or how many constant terms there may have been in the original. We indicate this usually by adding the letter C to indicate what is known as a "constant of integration." The C simply represents any constant terms that may have been in the original expression.

Thus, properly stated, $\int 3x^2 dx = x^3 + C$.

You will find, when you get further into your school course (or any good textbook on calculus), that when using integral calculus to solve applied problems in physics, chemistry and engineering there often are clues (in the nature of the problems themselves) as to the nature of these constants. Thus we often are able to evaluate them.

To help you become familiar with some of the symbology your textbook may use, we will use F(x) for the time being instead of y' to represent the derivative. Using these symbols we can then state the following general rule:

If $\frac{d}{dx}F(x) = f(x)$, then $\int f(x)dx = F(x) + C$.

Thus if $\frac{d}{dx}(x^3) = 3x^2$, then $\int 3x^2 dx = x^3 + C$.

Try this below for the function $F(x) = x^4$.

If _____ = _____, then _____ = _____.

Turn to page 90 to check your answers.

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To test this theory let's start with such an integrand, take its derivative, and see if we do in fact end up with the expression x^n. Thus,

$$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = \frac{n+1}{n+1}x^{n+1-1} = x^n.$$

We will not attempt -- nor need you attempt -- to perform this kind of reasoning about the other integration formulas. We went through the exercise above in order to give you some notion of the kind of reasoning that someone else had to go through in deriving the various integration formulas with which we work in integral calculus. If nothing else it should give you a healthy respect for their efforts. However, it also should provide you with some insight into the basic method of integrating functions. And it is something your classroom instructor will probably insist that you know.

From what we have covered above, we can now state the required integration formula:

I-1 $$\boxed{\int x^n dx = \frac{1}{n+1}x^{n+1} + C.}$$

With this formula in front of you, perform the integrations called for below, for the values of x^n shown:

1. When $x^n = 3x^2$, $\int x^n dx =$ _____ $n = 2$
2. When $x^n = 7x^6$, $\int x^n dx =$ _____ $n = 6$
3. When $x^n = 2x$, $\int x^n dx =$ _____ $n = 1$
4. When the integrand = 1, $\int dx =$ _____ $n = 0$

Turn to page 91 to check your answers.

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ans. If $\frac{d}{dx}(x^4) = 4x^3$, then $4x^3dx = x^4 + C$.

Most integration, like most differentiation, is done by formula. Since "working backward" as we have to do in integration is considerably more difficult, generally speaking, than performing the original differentiation, there are many more possible outcomes of the integration process hence many more tables of integrals, or integration formulas, than there are tables of derivatives. In fact, integral tables usually are published as a separate book in themselves. But to use such tables one needs to know how the formulas were arrived at.

Unfortunately, there is no uniform, step-by-step process of integration such as there is in differentiation. In general, integration is a process which has to be performed literally by thinking backwards!

For example, to find the formula for $\int x^n dx$ we first ask ourselves: What is the function which, when differentiated, yields x^n as its derivative? Recalling from differential calculus that $\frac{d}{dx}(x^n) = nx^{n-1}$, it is apparent that a formula somewhat like the one we are looking for would be

$$\int nx^{n-1}dx = x^n + C.$$

Although obviously not the answer to our question, this equation gives us a clue that if we had started with a power of x one degree higher, and if we had divided that power of x by its exponent, we would have had the desired integrand.

Continue on page 89.

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ans. 1. $x^3 + C$ 2. $x^7 + C$ 3. $x^2 + C$ 4. $x + C$

Did you get that last problem correct? It was intended challenge your thinking a bit and to lead you toward what is really a special case of the general rule for integrating powers of x which we developed on the previous page. Namely: The integral of 1 (written as dx with the 1 understood) is x plus a constant. Thus,

I-2 $$\boxed{\int dx = x + C.}$$

And, as you might expect, the integral of 0 is a constant. Thus,

I-3 $$\boxed{\int 0\,dx = C}$$

So far then we have three integration formulas:

I-1 The integral of a power of x: $\int x^n dx = \frac{1}{n+1} x^{n+1} + C$

I-2 The integral of 1: $\int dx = x + C$

I-3 The integral of 0: $\int 0\,dx = C$

On page 89 you practised using Rule I-1. However, since there are no variations of the situations covered by Rules I-2 and I-3 (i.e. either you have a 1 or a zero or you don't), there really is nothing to practice. Just watch out for these situations; be alert to recognize them and to apply the proper rules.

Now proceed to the next page and we will consider the situation where we need to find the integral of a constant times a variable.

ontinue on page 92.

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The integral of a constant, c, times a variable, v, is the same as the constant times the integral of the variable. Thus,

I-4 $$\boxed{\int cv\,dx = c\int v\,dx}$$

In other words, all we need do in this situation is bring the constant outside the integral sign and then proceed to integrate the term containing the variable. For example,

$$\int 4x^2dx = 4\int x^2dx = \tfrac{4}{3}x^3 + C.$$

And, as you might expect from the above formula, the integral of a constant is given by the formula:

I-5 $$\boxed{\int c\,dx = cx + C.}$$

This formula simply tells us again that the constant can be brought outside the integral sign until the integration has been performed, then brought back into the term as a multiplier. Thus,

$$\int 8\,dx = 8\int dx = 8x + C.$$

There is little more to be said about either of these integral formulas except to urge you to start memorizing them. You will use them frequently in integral calculus. They are sure to appear in one form or another on the tests your classroom teacher gives. Learning the basic differential and integral formulas is about like learning the multiplication tables in arithmetic; both are essential if you expect to get very far in your study of mathematics.

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Now let's consider the case where we have <u>two</u> variables, both of which are, in turn, variables of x. The rule for integrating in this case is: <u>The integral of the sum (or difference) of two variables, u and v, is the sum (or difference) of the integrals of the functions separately.</u>

Since this probably sounds like double-talk to you, let's write it down in symbols at once and the meaning will, I am sure, become clearer:

I-6 $$\boxed{\int (u + v)dx = \int u\,dx + \int v\,dx}$$

For example, $$\int (x^3 + x^4)dx = \int x^3 dx + \int x^4 dx$$

$$= \frac{x^4}{4} + \frac{x^5}{5} + C.$$

So in addition to the three integral formulas summarized on page 91 we now have three additional formulas, namely:

I-4 The integral of a constant times a variable: $\int cv\,dx = c\int v\,dx$

I-5 The integral of a constant: $\int c\,dx = cx + C$

I-6 The integral of the sum of two variables: $\int (u + v)dx = \int u\,dx + \int v\,dx$

Use all six formulas, as needed, to perform the following integrations:

1. $\int x\,dx =$ _____
2. $\int 2\,dx =$ _____
3. $\int 0\,dx =$ _____
4. $\int \frac{3}{2}x^2 dx =$ _____
5. $\int 2x^3 dx =$ _____
6. $\int (x^2 - 3x^3)dx =$ _____

Turn to page 94 to check your answers.

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ans. 1. $\frac{x^2}{2} + C$ 3. C 5. $\frac{1}{5}x^4 + C$

2. $2x + C$ 4. $\frac{x^3}{2} + C$ 6. $\frac{x^3}{3} - \frac{3x^1}{4} + C$

Let's list here the six integration formulas we have discussed so far so that you will have them all in front of you in one place:

I-1 $$\int x^n dx = \frac{1}{n+1} x^{n+1} + C$$

I-2 $$\int dx = x + C$$

I-3 $$\int 0\, dx = C$$

I-4 $$\int cv\, dx = c \int v\, dx$$

I-5 $$\int c\, dx = cx + C$$

I-6 $$\int (u+v)dx = \int u\, dx + \int v\, dx$$

Keep in mind that we have obtained all the above formulas by the fairly simple process of thinking backwards from what we already know about the original functions from which each was derived.

Keep in mind also that it is one of the (at times) frustrating facts of integral calculus that <u>you can't integrate anything before the reverse process of differentiating something else has yielded the expression you wish to integrate</u>!

This is why there are, indeed, some functions we <u>can't</u> integrate because no one has yet been able to find the expressions from which they were derived!

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So far the integration formulas we have worked with have all been of the form ...dx; that is, they have all dealt with fairly simple, direct functions of x. However, we often run into the situation where we need to be able to integrate the function of a function. Thus, if y is a function of v, and v is a function of x, we say that y is a function of a function. Although the explanation may be new to you, the idea should not be entirely novel since we have used the letters u and v to represent functions of x before.

Thus, corresponding to the formula $\int x^n dx = \frac{1}{n+1} x^{n+1} + C$, we have the formula:

I-7 $$\boxed{\int v^n dv = \frac{1}{n+1} v^{n+1} + C.}$$

Since v^n represents some function of x, what this formula is saying to us is that to be able to integrate such a function we must have the derivative of v with respect to x to start with! That is, we must have dv.

For example, suppose we wished to integrate the expression $2x(x^2 - a^2)^2$. Formula I-7 tells us we cannot do so unless we have the derivative of v in the integrand. Now v in this case is $(x^2 - a^2)$, hence $dv = 2x$ and — lo and behold! -- we do have 2x as part of the integrand, which means we can perform the integration. Accordingly, from formula I-7,

$$\int 2x(x^2 - a^2)^2 dx = \frac{1}{2+1}(x^2 - a^2)^{2+1} + C$$

$$= \tfrac{1}{3}(x^2 - a^2)^3 + C.$$

Find: $\int 9x^2(x^3 + 7)^2 dx =$ _____

Turn to page 96 to check your answer.

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ans. $(x^3 + 7)^3 + C$

Let's go over that last problem together.

You were asked to find: $\int 9x^2(x^3 + 7)^2 dx$.

The first thing we need to check on is whether or not we have dv -- that is, the derivative of $(x^3 + 7)$ with respect to x -- in the integrand. Do we? Yes, since the derivative of $(x^3 + 7)$ is $3x^2$.

While it is true that what we actually have is $9x^2$, this can be easily adjusted by dividing $9x^2$ by 3, making it $3x^2$, and offsetting this division by placing a 3 outside the integral sign as a multiplier. This gives us

$$3\int 3x^2(x^3 + 7)^2 dx.$$

Now, integrating according to formula I-7 we get

$$3 \cdot \frac{1}{3}(x^3 + 7)^3 + C$$

or simply $(x^3 + 7)^3 + C$.

There are many, many tricks to integrating and you will learn a great number of them in your regular course -- enough to convince you that integration is almost as much an empirical art as it is an exact science!

> Note: The foregoing problem, as well as the example that preceded it, could, of course, have been solved simply by expanding the integrand, integrating each term and then adding the results. The purpose in using the integration by parts technique was to give you practise in its application.

Now turn to the next page and we will look at one more trick.

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Since we have already found that the derivative of the sine is the cosine, it probably will not surprise you to learn that the integral of the cosine is (minus) the sine. Thus,

I-8 $$\boxed{\int \cos v\, dv = \sin v + C, \text{ and } \int \sin v\, dv = -\cos v + C.}$$

Apart from identifying this as integration formula I-8 there is little more to say about it -- except to caution you to remember it when you need it. And you will need it in the example below.

Now for the trick we spoke of.

There is no direct way of integrating the product of two functions of x, such as u and v. However, formula D-7 from page 52 enabled us to <u>differentiate</u> their product as follows:

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

and integration of both sides gives us: $uv = \int u\frac{dv}{dx} + \int v\frac{du}{dx}$. Rearranging the terms of this equation to read $\int v\frac{du}{dx} = uv - \int u\frac{dv}{dx}$ and dropping dx in the denominators (since it is implicit in the expressions du and dv) we get

I-9 $$\boxed{\int v\, du = uv - \int u\, dv.}$$

This may not impress you as much of a mathematical triumph, but it oftan turns out that u dv may be found directly from basic integration formulas, even though v du may not. This trick is called "integration by parts." Here's how it works.

<u>Example</u>: Find $\int x \cos x\, dx$. Here v = x, du = cos xdx, hence u (the integral of du) = sin x (from what we learned above), and dv = dx. Therefore,

$$\begin{aligned}\int x \cos x\, dx &= \int v\, du \\ &= uv - \int u\, dv \\ &= x \sin x - \int \sin x\, dx \\ &= x \sin x + \cos x + C.\end{aligned}$$

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Integration by parts will become meaningful to you and properly appreciated only after you have used the technique to solve a number of problems that would be difficult or impossible to handle in any other way.

You will have plenty of opportunities to apply this method in connection with the exercises in your classroom text, so we will not attempt to work with it further now. At least it should look familiar to you when you next encounter it.

Speaking of looking familar, does the integrand in the following expression seem familiar? Can you identify it?

$$\int \frac{1}{v}\,dv = \;?$$

$\frac{1}{v}\,dv$ represents _____

_____

Turn to page 100 to check your answer.

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Table of Integration Formulas

I-1 $$\int x^n dx = \frac{1}{n+1} x^{n+1} + C$$

I-2 $$\int dx = x + C$$

I-3 $$\int 0\, dx = C$$

I-4 $$\int cv\, dx = c \int v\, dx + C$$

I-5 $$\int c\, dx = cx + C$$

I-6 $$\int (u+v) dx = \int u\, dx + \int v\, dx + C$$

I-7 $$\int v^n dv = \frac{1}{n+1} v^{n+1} + C$$

I-8 $$\int \cos v\, dv = \sin v + C, \text{ and } \int \sin v\, dv = -\cos v + C$$

I-9 $$\int v\, du = uv - \int u\, dv + C$$

I-10 $$\int \frac{dv}{v} = \log_e v + C$$

And now it is time we looked at some applications of integral calculus. In the next chapter we will consider the distinction between definite integrals and indefinite integrals and see how the process of integration can be used to find, for example, the area under a curve.

Turn to page 101.

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ans. $\frac{1}{v}dv$ represents the derivative of the logarithmic function v to the base e; that is, $\frac{d}{dx}(\log_e v) = \frac{1}{v}\frac{dv}{dx}$ or $\frac{dv}{v}$.

Perhaps you remembered the log function from our discussion on page 71 of how to find the derivative of a logarithmic function. In any case it is another perfect example of the fact that we cannot integrate an expression unless we first are able to recognize it as the derivative of some other specific function!

Here is the complete integration formula for a logarithmic function:

I-10 $$\boxed{\int \frac{dv}{v} = \log_e v + C}$$

If you get the impression that we are moving much more rapidly in deriving our formulas for integral calculus than we did for differential calculus, you are quite right. The reason for this is, as we have mentioned before, that we cannot derive integration formulas as we do differential formulas by the delta method. Therefore, we can only familiarize ourselves with the integration formulas developed by others (research matematicians) over a period of many years. Our job is simply to recognize when and how to use them!

Now if you will turn to the next page you will find a list of the ten integration formulas we have discussed. These are, of course, only the most elementary ones, but the only ones you need be concerned with in this book.

Continue on page 99.

CHAPTER 7: APPLIED INTEGRATION

The type of integrals discussed in the last chapter are known as indefinite integrals. This is because they are of the general form $f(x)dx = F(x) + C$, (see page 89) hence no matter what value we substitute for x the value of the integral is still indefinite, since the constant of integration, C, can have any arbitrary fixed value. Hence the table of integrals appearing on page 99 actually is a table of indefinite integrals.

However, as suggested earlier, in specific applications the value of an indefinite integral can always be found by determining its constant of integration under the specific conditions of any given problem.

This leads us to the concept of the definite integral, and it is definite integrals we are going to discuss in this chapter. Whereas the indefinite integral is a function obtained by working backward from its derivative, the value of the definite integral is a number, defined by a limiting process, as we shall soon see.

The indefinite integral is then, in a sense, the link between the derivative and the definite integral.

To make sure you are clear as to the distinction between these two types of integral, indicate whether you consider the following statement true or false.

An integral is indefinite so long as it contains a constant that cannot be defined. True or False?_____

Turn to page 102 to check your answer.

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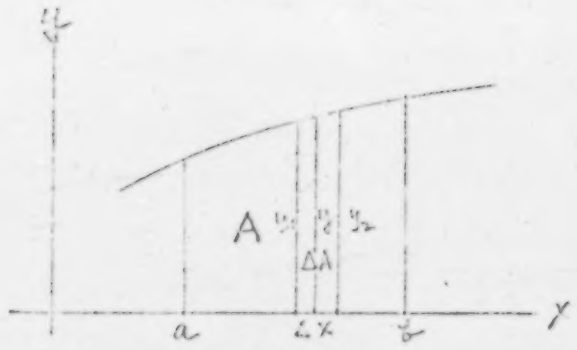
ans. True



We mentioned quite some time ago that one of the problems early mathematicians found difficult (impossible, actually, in a precise sense) to solve was that of finding the area under a curve. Integral calculus lends itself perfectly to the solution of this kind of problem. To find out why, consider the illustration above.

Suppose we wish to find the area A under the curve -- that is, between the curve and the x-axis -- and between the limits x = a and x = b. Let's start by considering a very small part of that area, namely, the segment shown above as being Δx wide and y high.

Now if we can assume for the moment that y is the average height of that small segment (that is, the average of y_1 and y_2), then the area of that tiny rectangle would be given by

$$\Delta A = y \cdot \Delta x.$$

Note: y isn't really the average height nor ΔA a rectangle. However, Δx is very small, and as we let it approach zero -- which we will in a moment -- y becomes the height and ΔA becomes a rectangle.

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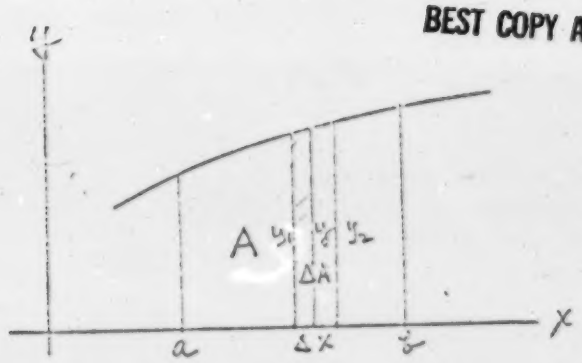

We found the area of ΔA -- that tiny portion of the whole area, A -- just as we would in any rectangle, by multiplying the base times the height. Once more, then,

$$\Delta A = y \cdot \Delta x,$$

which we can also write as $\frac{\Delta A}{\Delta x} = y$.

Now if we take the limit of this function as $x \longrightarrow 0$, and y_1 and $y_2 \longrightarrow y$, we get: $\lim_{\Delta x \longrightarrow 0} \frac{\Delta A}{\Delta x} = \frac{dA}{dx}$*, or $dA = y\,dx$.

Let's look at that last equation for a moment because it is most important to what we are doing and therefore most important that you are clear as to what it <u>says</u>.

<u>$dA = y\,dx$ says that each little bit of the area under the curve is equal to the width of that little bit, Δx, multiplied by the height of the curve, y, at each successive point along the curve</u>.

This tells us that if we add up <u>all</u> the little slices of area under the curve we should arrive at the <u>total</u> area of A.

Have you any idea how we will go about adding them up? If you have, write it down here:_____

---

*Turn to page 38 to review the relationship between a ratio such as $\frac{y}{x}$ and the derivative $\frac{dy}{dx}$, in case you have forgotten.

To check your answer turn to page 104.

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ans. Integrate both sides of the equation.

That's right: Integrate both sides of the equation.

Thus $\int dA = \int y \cdot dx$, or $A = \int y \cdot dx$.

Integration basically is a summing up process! That is, the desired quantity is obtained as the limit of the sum when the number of its parts is increased indefinitely. In this case we wish to sum up all the small bits of area to get the total area A.

Since y simply represents some function of x, we can replace it with the more general expression f(x) as we frequently have done before. They mean the same thing. Thus we can write $A = \int f(x)dx$.

We are not through yet because we still have not actually evaluated the integral on the right-hand side for the specific limits x = z and x = b, which make it a definite integral. (Remember we set out to find the area between these two limits.)

It is customary to show these limits in this way: $\int_a^b f(x)dx$, assuming that a is smaller than b. This is read "the integral from a to b of y dx." We are not going to burden you with the proof, but it turns out that

$$\int_a^b f(x)dx = F(b) - F(a).$$

Or, putting it in words: Integrate the differential of the area under the curve and substitute in this, first the upper limit and then the lower limit for the variable, and subtract the last result from the first.

The constant of integration, you will note, disappears in the subtraction and therefore need not be considered.

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As is often the case, you will find evaluating definite integrals easier to do than to read about. But before we start let's write down our formula again so that we will have it right in front of us when performing the evaluation.

$$\int_a^b f(x)dx = F(b) - F(a)$$

Example: Find the value of the definite integral

$$\int_0^2 3x^3 dx$$

Step 1. Integrate $\left[\frac{3x^4}{4}\right]_0^2$

Step 2. Substitute upper limit: $F(b) = \left[\frac{3(2)^4}{4}\right] = 12$

Step 3. Substitute lower limit: $F(a) = \left[\frac{3(0)^4}{4}\right] = 0$

Step 4. Subtract the last from the first:

$F(b) - F(a) = 12 - 0 = 12$

Hence the value of the integral (area under the curve) is 12.

Example: Find the value of the definite integral

$$\int_0^2 (4x - x^3)dx$$

Step 1. Integrating (from formula I-6, page 93): $\left[2x^2 - \frac{x^4}{4}\right]_0^2$

Step 2. Substituting (here we will combine steps 2, 3, and 4 above):

$$\left[2(2)^2 - \frac{2^4}{4}\right] - \left[2(0)^2 - \frac{0^4}{4}\right]$$

or $8 - 4 - 0 = 8 - 4 = 4.$

Get the idea? Now here's one you can try on your own. Find the value of the definite integral $\int_0^3 4x^3 - x^2$.

Turn to page 106 to check your answer.

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ans. 72

Did you get the correct answer? Here's the solution just in case you had any trouble.

We start with the definite integral $\int_0^3 4x^3 - x^2$.

Integrating we get: $\left[\frac{4x^4}{4} - \frac{x^3}{3}\right]_0^3$

And substituting limits: $\left[\frac{4(3)^4}{4} - \frac{3^3}{3}\right] - \left[0\right]$

Which gives us our answer: $81 - 9 = 72.$

Now it is time to apply this procedure to finding the area between two ordinates under some recognizable curve so that you will see just how the whole concept works in an applied way.

So look at the next page and we will consider the problem of finding the area under the parabola $y = x^2$, both with relation to the x-axis and the y-axis, just for practice.

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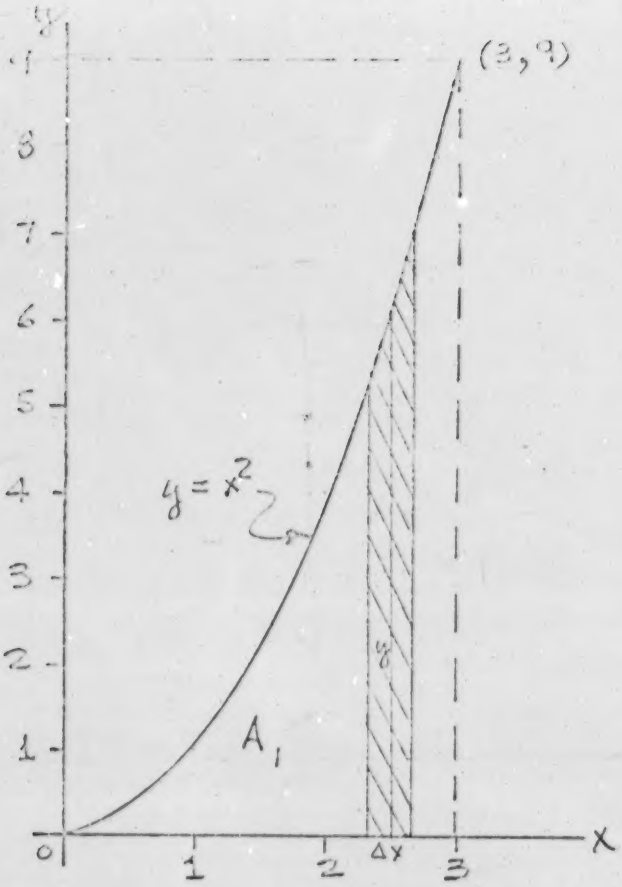

Example 1: Find the area A under the curve $y = x^2$ between the limits $x = 0$ and $x = 3$.

($y = x^2$ was chosen for $f(x)$ in this example because it easy to integrate and has some other advantages we will discuss later.)

Definite integral: $A_1 = \int_0^3 x^2 dx$

Integrating $x^2 dx$ we get: $\left[\frac{x^3}{3}\right]_0^3$

From which: $A_1 = \left[\frac{3^3}{3}\right] - \left[\frac{0^3}{3}\right]$

Or: $A_1 = \left[\frac{27}{3}\right] - [0] = 9.$

Nine, then (in whatever units), represents the area under the curve $y = x^2$ and the x-axis between the ordinates $x = 3$ and $x = 0$. Now let's consider the area between the same curve and the y-axis.

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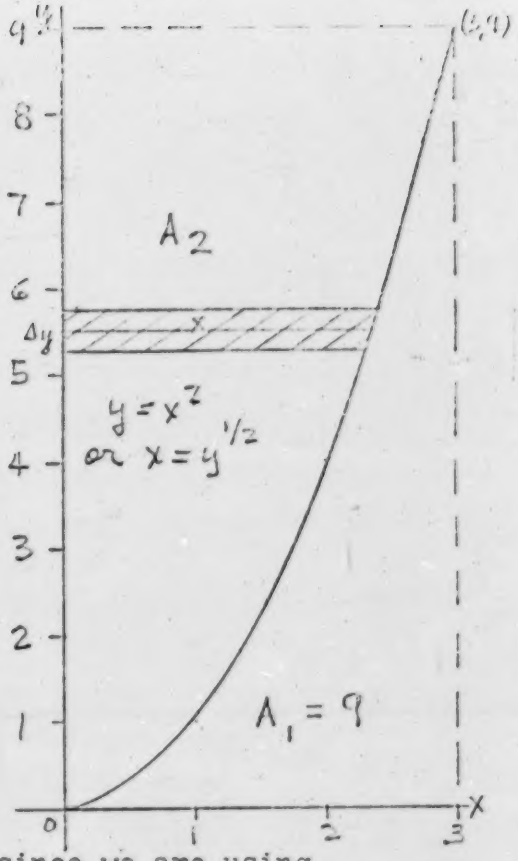

Example 2: You recall we said that the function $y = x^2$ had some other advantages? One of them is that we can easily check the results of our integration. How? It is apparent that the entire area, A, of the rectangle bounded by the x-axis, the y-axis, and the limits x = 3, y = 9 is 3·9 or 27. And since we just found $A_1 = 9$ then, since $A = A_1 + A_2$, A_2 should equal 18. Let's see if it does.

This time our A is equal to x dy, since we are using a horizontal instead of a vertical segment. Hence we can write

$$A_2 = \int_0^9 x\,dy$$

or, since $x = y^{\frac{1}{2}}$,

$$A_2 = \int_0^9 y^{\frac{1}{2}}\,dy$$

Integrating $y^{\frac{1}{2}}dy$ we get:

$$\left[\frac{2}{3}y^{3/2}\right]_0^9 = \left[\frac{2}{3}\cdot 27\right] - \left[\frac{3}{2}\cdot 0\right]$$

Or:

$$A_2 = 18$$

Hence $A_1 + A_2 = 27$, and we have verified our results.

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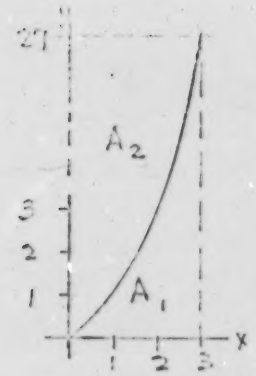

$$y = x^3$$

or $$x = y^{1/3}$$

Now it's your turn.

Above is the graph of a slightly steeper exponential curve whose shape isg given by the function $y = x^3$, considering x as the independent variable. However, if we think of y as the independent variable then, taking the cube root of both sides of the equation, we get $x = y^{1/3}$. You will need both these expressions, just as we needed them in our two examples on the preceding pages.

Problem: Prove that the total area A -- made up of the area between the curve and the x-axis, A_1, and the area between the curve and the y-axis, A_2 -- is 81. (To do this you will integrate the function $y = x^3$ between the limits $x = 0$ and $x = 3$, then integrate the function $x = y^{1/3}$ for the limits $y = 0$ and $y = 27$, then add your results together.)

ans. A_1 = _____ = _____ = _____

A_2 = _____ = _____ = _____

A_1 _____ + A_2 _____ = _____

Turn to page 110 to check your answers

From page 109

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ans. $A_1 = \int_0^3 x^3 dx = \left[\frac{x^4}{4}\right]_0^3 = \frac{81}{4}$

$A_2 = \int_0^{27} y^{1/3} dy = \left[\frac{3}{4}y^{4/3}\right]_0^{27} = \frac{243}{4}$

$A_1 \frac{81}{4} + A_2 \frac{243}{4} = \frac{324}{4} = 81$

At this point you might properly expect a few more examples or problems showing further applications of the definite integral. If so, you will be disappointed, for this as a far as we shall go with integration.

The reason for stopping here is that although there are many examples that we might work together showing the use of integration to find areas under curves, the lengths of sectors, the volumes of many kinds of geometric solids, the centroids of bodies -- to say nothing of the applications to electricity and electronics -- nearly all of these would require the use of integral forms with which you are as yet still unfamiliar. And, alas, there is no time to examine them in this brief introduction to calculus. Hopefully, you will have time to do so in your regular course.

But now it is time instead that we review what we have covered with regard to integral calculus.

Accordingly, if you will turn to page 111 we will begin our summary of this aspect of the subject and follow this with a brief self-quiz that will enable you to see how well you have done -- and perhaps be guided to review as necessary.

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Review of integral calculus concepts

Review Item	Page ref.	Example
1. Integral calculus is the inverse of differential calculus.	85, 86	$y = x^3$ $y' = 3x^2$ $3x^2dx = x^3 + C$
2. The integral symbol is an elongated S.	86	$\int$
3. The expression that appears immediately after the integral sign is called the integrand, that is, the function to be integrated.	86	$3x^2$ integrand
4. When differentiating an expression, any constants in it drop out. Therefore, not knowing what constants may have been in it originally, when we integrate a function we must always add the letter C to represent the "constant of integration," a collective term for whatever constants may have been in the original expression.	88	$\int x^3 = \frac{x^4}{4} + C$
5. The symbol F(x) often is used in integral calculus to represent the original (undifferentiated) function as an alternate to y. Similarly, f(x) often serves in place of y' to represent the derivative.	88	$y = x^4$ $y' = 4x^3 = f(x)$ $\int f(x) = \int 4x^3 = x^4 + C = F(x)$
6. Integration is, in general, a process that has to be performed by thinking backwards.	90	If $\frac{d}{dx}(x^n) = n\,x^{n-1}$ then $\int nx^{n-1} = x^n + C$
7. To integrate a simple exponential function whose original form we don't know, we use the formula: $x^n dx = \frac{1}{n+1}x^{n+1}$	89	If $f(x) = 3x^3$ then $\int 3x^3 dx = \frac{3x^{3+1}}{3+1}$ $= \frac{3x^4}{4} + C$

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Review of integral calculus (continued)

Review Item	Page ref.	Example
8. The integral of 1 (written as dx with the 1 understood) is equal to x plus a constant.	91	$\int dx = x + C$
9. The integral of 0 (zero) is a constant.	91	$\int 0\,dx = C$
10. The integral of a constant times a variable is the same as the constant times the integral of the variable.	92	$\int cv\,dx = c\int v\,dx$ $\int 2x\,dx = 2\int x\,dx = x^2 + C$
11. The integral of a constant is given by the formula: $\int c\,dx = cx + C$	92	$\int 3\,dx = 3x + C$
12. The integral of the sum (or difference) of two variables, u and v, is simply the sum (or difference) of the integrals of the functions separately. This rule tells us how to add or subtract integrals. Thus, $\int(u+v)dx = \int u\,dx + \int v\,dx$	93	$\int(x^2 + x^3)dx = \int x^2dx + \int x^3dx$ $= \frac{x^3}{3} + \frac{x^4}{4} + C$
13. It is possible to integrate the function (often represented by the letter v) of x. For example, the integration of v to some power is quite similar to the integration of x to a power. It requires, however, that we have the derivative of v with respect to x in the integrand to start with! $\int v^n dv = \frac{1}{n+1}v^{n+1} + C$	95	dv means $\frac{dv}{dx}dx$ $\int 3x^2(x^3 + b^3)^2dx = ?$ $v = (x^3 + b^3)$ $dv = 3x^3$ hence $\int 3x^2(x^3+b^3)^2dx =$ $\frac{(x^3 + b^3)^3}{3} + C.$

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Review of integral calculus (continued)

Review Item	Page ref.	Example
14. When confronted with the task of integrating an expression involving two functions of x (which we designate as u and v) we use the following formula: $\int v\,du = uv - \int u\,dv$	97	Find $\int x \sin x\,dx$ Here $v = x$, $du = \sin x\,dx$, $dv = dx$, and $u = \int \sin x\,dx = -\cos x$ hence: $\int x \sin x\,dx = -x\cos x + \int \cos x\,dx$ $= -x\cos x + \sin x + C.$
15. Remembering that the derivative of the logarithm of v to the base e is $\frac{1}{v}\frac{dv}{dx}$ or just $\frac{dv}{v}$, we recognize correspondingly that $\int \frac{dv}{v} = \log_e v + C$	100	Find: $\int \frac{x^2}{1-x^3}\,dx.$ Here $v = (1 - x^3)$ and $dv = -3x^2$, hence $\frac{x^2}{1 - x^3}\,dx =$ $-\frac{1}{3}\int \frac{-3x^2}{1 - x^3}\,dx$ (multiplying numerator and denominator by -3 and bringing the -3 in the denominator outside the integral sign) or $= -\frac{1}{3}\log_e(1-x^3) + C.$
16. The value of an indefinite integral can always be found by determining its constant of integration under the specific conditions of any given problem.	101	
17. The value of the definite integral is a number, defined by a limiting process.	101	
18. Integration, being basically a summing process, it can be used to find the area under a curve.	102, 103	(See pages 102-109)

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Review of integral calculus (continued)

Review Item	Page ref.	Example
19. To find the value of a definite integral between two given limits, a and b, perform the integration and substitute in this, first the upper limit and then the lower limit for the variable, and subtract the last result from the first. $\int_a^b f(x)dx = F(b) - F(a)$	104, 105	$\int_1^2 4x^3 dx = \left[x^4\right]_1^2$ $= \left[2^4\right] - \left[1^4\right]$ $= 16 - 1$ $= 15$
20. Finding the area under a curve is accomplished by means of the procedure shown in item 19 above.	107, 108	(See examples on pages 107 and 108)

And now, having completed the review, you should be curious to see how much you have understood and retained on the subject of integral calculus -- to the extent of our brief introduction to the subject.

Turn to page 115 and you will find a short self-quiz that should help you answer this question.

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Self-Quiz on the Basic Concepts of Integral Calculus

(Circle the correct answer or fill in the missing information)

1. Find: $\int x^4 dx$ ans._____

2. Integration is the inverse of differentiation. True False

3. $\int_2^4 3x^2\,dx$ is a(n)_____. definite integral indefinite integral

4. What is the integral of zero dx ($\int 0\,dx$)? ans._____

5. Draw a circle around the integrand in the following expression: $\int 4x^3 dx$

6. Find: $\int dx$ ans._____

7. The following is a definite integral: $\int 2x(x^2 + 3)dx$. True False

8. In general, integration is a process that must be performed by thinking backwards. True False

9. Find: $\int 7x\,dx$ ans._____

10. Find: $\int (x - x^3)\,dx$ ans._____

11. The value of the definite integral is a number, defined by a limiting process. True False

12. The value of an indefinite integral can always be found by determining its constant of integration under the specific conditions of any given problem. True False

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Self-Quiz on Integral Calculus (continued)

13. Complete the following integration formula: $\int v^n dv =$ _____

14. $\int_0^3 x^3 dx = 20\frac{1}{4}$ True False

15. It is one of the facts of life of integral calculus that you can't integrate anything until the reverse process of differentiating something else has yielded the expression you want to integrate. True False

16. The following expression can be integrated just as it stands: True False

$$\int x(x^2 + 4)dx$$

17. In integrating an expression such as $\int v^n dv$ we must have the derivative of v with respect to x to start with. True False

(Does your answer to this problem agree with your answer to the previous problem?)

18. Find the numerical value of the following integral: $\int_2^4 x^2\, dx$. ans._____

19. Because integration is basically a _____ process it can be used to find the area under a curve.

evaluative
precise
summing

Self-Quiz on Integral Calculus (continued)

20. The constant of integration, C, is used to represent:

(a) The name of the mathematician, Clavius, who first developed this concept.

(b) The particular constant term that was in the original expression.

(c) Any constant terms (collectively) that may have been in the original expression.

To check your answers turn to page 118.

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Answers to Self-Quiz on Integral Calculus

		Page ref.
1.	$\frac{x^5}{5} + C$	86
2.	True	85
3.	definite integral , because it has stated limits	104
4.	C	91
5.	Your circle should be around: $4x^3dx$	86
6.	$x + C$	91
7.	False, because no limits are given	104
8.	True	90
9.	$\frac{7x^2}{2} + C$	92
10.	$\int(x - x^3)dx = \int x\,dx - \int x^3\,dx = \frac{x^2}{2} - \frac{x^4}{4} + C$	93
11.	True	101
12.	True	101
13.	$\int v^n dv = \frac{1}{n+1} v^{n+1} + C$	95
14.	True	105
15.	True	94
16.	False, because you don't have dv, the derivative of v with respect to x, which in this case would be 2x. Hence a 2 would have to be inserted before the x as a multiplier and a $\frac{1}{2}$ placed <u>in front</u> of the integral sign to compensate. In other words, the entire integral would have to be multiplied by $\frac{2}{2}$ before the integration could be performed.	95
17.	True (see above)	95
18.	$\frac{56}{3}$ or $18\frac{2}{3}$	105

	Page ref.
19. summing-up	104
20. Any constant terms (collectively) that may have been in the original expression.	88

Now that you have completed the two quizzes, check back and see how many answers you got correct altogether. If you got 36 correct (80%) you did well. If you got 40 correct you did very well.

In parting let me remind you once more that the entire emphasis of this book has been on introducing you to the very basic concepts on which differential and integral calculus are founded. While no attempt has been made to develop these concepts or to show their nearly limitless applications in both theoretical and applied mathematics, the author sincerely hopes that this brief exposure to the subject will at least have served to remove some of the mystery and terror that usually surround it. Also that it will help relieve some of the confusion and pressure that seem such an inevitable part of every first course in calculus.

If so, my purpose will have been achieved.

CHAPTER 8: HISTORICAL PERSPECTIVE

The word calculus comes from the Latin word meaning pebble, because in ancient times people used pebbles to count with. And even though the name has this historical connection with early mathematics, it has little logical connection with it, since calculus was developed in fairly recent times after much intervening growth in knowledge. The name given this branch of mathematics by one of its inventors (Newton) is descriptive of its field of application. He called it fluxions, referring to the fact that calculus deals with change. The subject today -- often referred to as "the calculus" -- is a body of rules for calculating with derivatives and integrals.

By 3000 B.C. the peoples of ancient Babylonia, China and Egypt had developed a practical system of mathematics. They used written symbols to stand for numbers and knew the simple arithmetic operations. They were able to apply their knowledge to government and business and had developed a practical geometry useful in engineering and agriculture. The ancient Egyptians knew how to survey their fields and how to make the intricate measurements necessary to build large pyramids. But this early mathematics was applied rather than pure. That is, it solved only practical problems.

The Greeks took the next major step in mathematics when, between 600 and 300 B.C., they became the first

people to separate mathematics from pratical problems. Geometry for the first time became an abstract exploration of space based upon a study of points, lines and figures such as triangles and circles. Interest in mathematics turned to logical reasoning rather than to facts found in nature. It became a blend of mathematics and philosophy, since the Greeks were mainly interested in geometry as a means of advancing logical reasoning and therefore developed the subject along this line.

Even at this early date, however, these "philosomaticians" ran into a number of puzzling problems. Some of these are embodied in the paradoxes of Zeno (495-435 B.C.). One involves a mythical race between Achilles and the tortoise. Even if the tortoise begins the race with a 100=yard start, if Achilles can run ten times as fast as the tortoise it seemed perfectly apparent that he would overtake the tortoise. The problem was to disprove Zeno's "proof" that the tortoise would always be ahead. He reasoned this way: while Achilles is covering the 100 yards that separates them at the start, the tortoise moves forward 10 yards; while Achilles dashes over this 10 yards, the tortoise plods on a yard and is still a yard ahead; when Achilles has covered <u>this</u> one yard, the tortoise is still 1/10th of a yard ahead. Thus, by dividing the distance run by Achilles into smaller and smaller amounts, Zeno argued that he would never pass the tortoise. The fact that an infinte set of distances could add up to a finite total distance was the unknown fact

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that made Zeno's "proof" appear plausible. It was not until a better understanding of limits was developed that it became possible to demonstrate the fallacy in Zeno's logic.

But there were other problems as well arising from this lack of a doctrine of limits. Most of these involved calculating the measures of curved figures: the area of a circle or of the surface of a sphere, the volume of a sphere or of a cone, and similar problems. Problems of this kind were treated by what came to be known as the Method of Exhaustions, actually a method of limits wherein the circle was regarded as a limit of a series of inscribed polygons. This method enabled Archimedes (287-212 B.C.) to arrive at very close approximations of the correct values in many cases.

A related method of limits, much more general in form, is one of the essential features of calculus today. Another problem, that of continuous motion, also was the subject of much speculation. The Greeks made important conceptual contributions toward an understanding of motion (partly under the pressure of Zeno's paradoxes,no doubt). But not until the development of calculus was there available a workable, systematic method for describing in both qualitative and quantitative terms such things as velocity and acceleration, and for making analytical studies of various particular motions.

Euclid, who lived about 325 B.C., was one of the foremost of the Greek mathematicians. It was he who left to posterity one of the greatest works of all time. His book, The Elements, is a summary and arrangement of all the mathematical knowledge of his age. It is of particular interest to us today because it contains most of the plane geometry taught in our present-day schools. And although Diophantus (c.A.D. 275) worked on numbers in equations, Greek mathematics was developed essentially without algebra. It was not until after the creation of analytic geometry in the 17th century that the way was opened for the advances in thought that marked the beginning of rapid progress in the study of motion and other types of continuous change.

Apollonius, who was known as the "Great Geometer," is believed to have lived during the period 260-200 B.C. His greatest contribution was to the study of sections cut from a cone by passing a plane through it. He called the resulting curves ellipses, hyperbolas and parabolas, just as we do today when we study them in plane analytic geometry, although our method of approach is quite different from that used by Apollonius.

After the fall of Rome in A.D. 476, Europe saw no new developments in mathematics for hundreds of years. The Arabs, however, preserved the mathematical tradition of the Greeks and Romans. Then, during the Middle Ages, one of the greatest discoveries in the history of mathematics

appeared when mathematicians in India developed zero and the decimal number system. After A.D. 700 the Arabs adopted these inventions from the Indians and used the new numbers in their mathematics. The Arabs also preserved and translated many of the great works of Greek mathematicians. After 1100, Europeans began to borrow the mathematics of the Arab world, including use of the decimal number system for business and to study Arab works on algebra and geometry.

Gradually interest in pure mathematics grew. During the 1500's much pioneering work was done in the development of algebra, including the use of letters to stand for unknown numbers. The basic concepts and procedures of trigonometry also were developed. Many more advances occurred in the 1600's, including the invention of logarithms and the development of new methods for algebra. A major event was the publication, by the brilliant French mathematician Rene Descartes in 1637, of the first work on analytic geometry, for the first time linking algebra and geometry in a precise way. The rectangular coordinate system we use today is called Cartesian in honor of Descartes, who used a modified form of our present coordinate system in his work.

Descartes' method, which in our present day terminology relates the distances of a point from two intersecting lines by means of an equation, opened the way for the advances in thought that marked the beginning of rapid progress in the study of motion and other types of continuous change.

Despite advances made in the field of mathematics from the time of Archimedes until that of Descartes -- advances in geometry, arithmetic, algebra, astronomy and dynamics -- the intervening centuries actually were among the least prolific in the history of mathematics. It was the creation of analytic geometry that finally made possible the appearance of a revolitionary new idea that was to unlock the door to an entire treasure house of new mathematics.

Calculus furnished that key.

It was Sir Isaac Newton (1642-1727) and Baron Gottfried Wilhelm von Leibnitz (1646-1716) who, working separately, invented the calculus independently of one another. It is another instance of the time being ripe for the development of an idea, and the idea came.

Calculus is a natural outgrowth of the application of algebra and analytic geometry to certain problems in physics and geometry. As we have seen, some of these problems had been considered by the mathematicians of ancient Greece. The nature of continuous motion -- how to analyze it and how to describe it accurately -- was one such problem, and the subject of much speculation. Although the Greeks did indeed make important conceptual contributions toward an understanding of motion, it was not until the development of calculus that there was available a workable, systematic method for describing such things as velocity and acceleration in qualitative and quantitative terms, and for making analytical studies of various particular motions. At first the basic concept of calculus --the underlying

idea of limits -- was seen only dimly. Not until nearly a century and a half later did the French mathematician A.L. Cauchy (1789-1857) give the doctrine of limits its final form, a doctrine that emerged clearly as the foundation for much of the structure of modern mathematics.

Newton was not only a powerful mathematician but also a scientist with a vivid and trained imagination. It was his interest in the motions of the sun, moon and planets, in tidal action and falling bodies -- culminating in his famous laws of gravity and motion -- that led to his need for some precsie, mathematical method for determining instantaneous velocity and for expressing the transient relationships between time, distance, velocity and acceleration, that is, a way to handle dynamic problems. The traditional problems of finding the tangent to a curve at a point, the area bounded by a closed curve, and the volume bounded by a surface also pressed for solution.

It is not surprising, therefore, that the two central concepts that finally emerged are, as we have seen from the foregoing chapters, interpretable in terms of motion and area. The one concept, that of the derivative, is illustrated by the velocity of a moving point. The other concept, that of the integral, is illustrated by the area of a certain geometric figure having a curved line as part of its boundary.

It would not be fair to leave you with the impression that calculus emerged full-blown with Newton and Leibnitz. Many of their concepts that emerged almost as inspiration had to be substantiated by rigid mathematical proof -- and this took time, and the work of many other brilliant mathematicians. Although the basic notions on which calculus rests have not changed to any great extent since Newton's day, the techniques, applications and extensions of these fundamental ideas have been expanded enormously.

Perhaps you yourself will some day add another chapter to the history of the development of calculus. Who knows?

# # #